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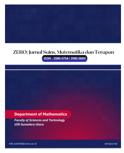

Exact Formulas for the Energy, Laplacian Energy, and Signless Laplacian Energy of Line Graphs of Lintang Graphs

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ABSTRACT

This study establishes exact formulas for the energy, Laplacian energy, and signless Laplacian energy of the lintang graph L_n and its line graph $L(L_n)$. An algebraic spectral framework is employed to construct the associated matrices and determine their eigenvalues explicitly. In contrast to general results on line graph energies introduced by Ivan Gutman and subsequent studies, this work focuses on a specific two-hub graph structure for which explicit spectral formulas have not been previously reported. The results show that $E(L_n) = 2\sqrt{2n}$, $\mathcal{L}E(L_1) = QE(L_1) = \frac{10}{3}$, $\mathcal{L}E(L_n) = QE(L_n) = \frac{4(n^2-n+2)}{n+2}$ for $n \geq 2$, and $E(L(L_n)) = \mathcal{L}E(L(L_n)) = QE(L(L_n)) = 4n-4$. These findings reveal a transition from sublinear to linear growth under the line graph transformation. The results contribute to spectral graph theory and are relevant to chemical graph theory and network analysis.

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1. INTRODUCTION

Graph theory plays a fundamental role in modern mathematics by providing a versatile framework for representing relationships and interactions within complex systems. A graph, consisting of vertices connected by edges, serves as an abstraction for modeling a wide range of real-world phenomena, including communication networks, social interactions, and molecular structures in chemistry [1], [2], [3], [4]. The arrangement of vertices and edges often reflects the complexity of the underlying system, making graphs effective tools for analyzing relational structures and system behavior [5], [6], [7], [8]. Beyond their combinatorial properties, graphs also possess spectral characteristics that can be studied through the eigenvalues of associated matrices, forming the foundation of spectral graph theory.

Among the various graph classes, the lintang graph L_n, defined as the join of $\overline{K_2}$ and $\overline{K_n}$ for $n \in \mathbb{N}$, represents a structured configuration consisting of two central vertices connected to a set of peripheral vertices

[9]. This construction produces heterogeneous connectivity and internal asymmetry, distinguishing it from more regular graph families. Such a two-hub structure naturally appears in simplified models of systems with dominant interacting components, for example in hub-based communication networks or molecular graphs with two principal interaction centers. Therefore, investigating its spectral properties provides insight into how structural imbalance influences global graph behavior.

Graph transformations offer another perspective for examining how structural modifications affect graph properties. One of the most widely studied transformations is the line graph construction, in which each edge of a graph is associated with a vertex in a new graph. The line graph of the lintang graph, denoted by $L(L_n)$, encodes edge-incidence relationships into adjacency relations among vertices [10]. This transformation is known to significantly modify spectral characteristics and has been extensively studied in the context of graph energy. Foundational work by Ivan Gutman and subsequent studies have explored the energy of line graphs and their relationships with the original graphs [10], [11], [12]. However, these results mainly address general graph families and do not provide explicit spectral formulas for structured graphs with heterogeneous vertex partitions such as lintang graphs.

Within this framework, graph energy and its variants, including Laplacian energy and signless Laplacian energy, serve as fundamental spectral invariants for characterizing structural properties of graphs. The concept of Laplacian energy was formally introduced by Gutman and Zhou [13], and further developed through comprehensive studies on graph energies and their mathematical properties [14]. These measures are defined through the eigenvalues of adjacency, Laplacian, and signless Laplacian matrices, providing quantitative insight into how connectivity is distributed within a graph.

Recent developments have demonstrated that these spectral measures play an important role in both theoretical and applied contexts. In particular, energy-based approaches have been utilized in network analysis, including structured networks such as torus graphs [15], while comparative studies have examined relationships among different types of graph energies [16]. Further advances have explored spectral radius, spread, and energy-related invariants, especially in the context of signless Laplacian matrices and their associated properties [17], [18], [19], [20], [21].

In addition, these concepts have been extended to more specialized graph classes, including graphs with self-loops, algebraic graph structures, and applications arising from real-world systems [22], [23], [24], [25], [26]. Other studies have focused on Laplacian spectra, matrix approximations, and energy in specific algebraic or combinatorial graph constructions, such as zero divisor graphs and hypergraphs [27], [28], [29]. In many of these cases, obtaining explicit expressions for spectral energies depends on exploiting the structure of the associated matrices. In particular, block matrix techniques have proven to be highly effective for graphs with non-uniform vertex partitions or heterogeneous connectivity patterns [8], [30].

Despite these developments, explicit formulas for the energy, Laplacian energy, and signless Laplacian energy of the lintang graph and its line graph have not yet been established in the existing literature. This gap is significant because the structural characteristics of lintang graphs lead to non-trivial spectral patterns that cannot be directly inferred from general line graph results. Moreover, understanding these spectral properties is essential for explaining how graph transformations influence energy growth and spectral invariants.

Motivated by this gap, this study derives exact formulas for the energy, Laplacian energy, and signless Laplacian energy of both L_n and $L(L_n)$ using an algebraic spectral approach based on block matrix decomposition. The results provide explicit spectral characterizations and reveal that the transformation from L_n to $L(L_n)$ changes the energy growth from sublinear to linear behavior. These findings contribute to the development of spectral invariants for structured graph classes and offer potential relevance in applications involving hub-based interaction models.

The remainder of this paper is organized as follows. Section 2 describes the research methodology and spectral framework employed in this study. Section 3 presents the main results and analysis, including the derivation of eigenvalues and exact energy formulas. The final section summarizes the main findings and outlines directions for future research.

2. RESEARCH METHOD

This study adopts an algebraic–spectral approach to investigate energy-related invariants of the lintang graph and its associated line graph. The analysis begins with a structural characterization of the lintang graph L_n, including its order, size, and vertex partition. These structural properties serve as the foundation for constructing the adjacency matrix, degree matrix, Laplacian matrix, and signless Laplacian matrix. Since the eigenvalues of these matrices determine graph energy, Laplacian energy, and signless Laplacian energy, their explicit formulation plays a central role in the analysis.

Throughout this study, standard matrix notation is employed. The identity matrix of order k, denoted by I_k, is defined as a square matrix whose diagonal entries are equal to 1, while all other entries are equal to 0. In addition, J_k denotes the $k \times k$ all-ones matrix, that is, a matrix in which every entry is equal to 1. These matrices are used to represent regular block structures that arise naturally in the spectral analysis. All graphs considered are simple and undirected. The terminology is used consistently, where the lintang graph is denoted by L_n and its line graph by $L(L_n)$. The lintang graph considered in this work is defined as follows.

Definition 1 [9] The lintang graph L_n is defined as the join of $\overline{K_2}$ and $\overline{K_n}$ for $n \in \mathbb{N}$. Its vertex set is

$$V(L_n) = \{ v_w, t_i \mid 1 \leq w \leq 2,\ 1 \leq i \leq n \}, \tag{1}$$

and its edge set is

$$E(L_n) = \{ v_w t_i \mid 1 \leq w \leq 2,\ 1 \leq i \leq n \}. \tag{2}$$

This construction yields a graph in which each vertex v_w of $\overline{K_2}$ is adjacent to every vertex t_i of $\overline{K_n}$, forming the characteristic two-hub structure that distinguishes the lintang graph.

To investigate how structural transformations affect spectral properties, this study also considers the line graph of the lintang graph.

Definition 2 [10] The line graph of the lintang graph, denoted $L(L_n)$, is defined by assigning a vertex to each edge of L_n, so that $V(L(L_n)) = E(L_n)$. Two vertices $x, y \in V(L(L_n))$ are adjacent if and only if their corresponding edges in L_n share a common vertex. This construction encodes the edge incidence relations of L_n into a new graph whose connectivity is governed entirely by the presence of a common vertex in the original structure.

For illustrative purposes, the structures of L_3 and $L(L_3)$ are presented to demonstrate how edge-incidence relations in L_n are transformed into vertex adjacency in $L(L_n)$.

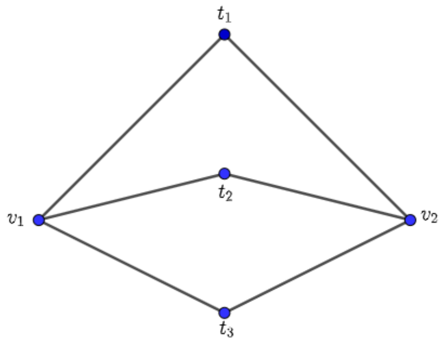

Figure 1. Lintang Graph L_3

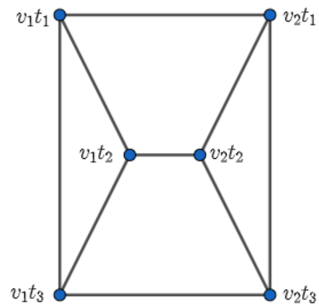

Figure 2. Line Graph of Lintang Graph $L(L_3)$

The spectral analysis employed in this study relies on several standard concepts in spectral graph theory.

Definition 3 [21] Let G be a graph. The spectrum of G is represented as a $2 \times j$ matrix, with the first row comprising the eigenvalues of the associated matrix of G and the second row indicating the corresponding multiplicities of each eigenvalue.

Definition 4 [10] Let G be a graph with adjacency matrix $A(G)$. The energy of G, denoted by $E(G)$, is formally defined as

$$E(G) = \sum_{i=1}^{n} |\gamma_i|, \tag{3}$$

where $\gamma_1, \gamma_2, \dots, \gamma_n$ are the eigenvalues of $A(G)$.

Definition 5 [10] Let G be a graph of order n and size m, with adjacency matrix $A(G)$ and degree matrix $D(G)$. The Laplacian matrix of G is defined as $\mathcal{L}(G) = D(G) - A(G)$, while the signless Laplacian matrix is defined as $Q(G) = D(G) + A(G)$. Let the eigenvalues of $\mathcal{L}(G)$ be denoted by $s_1, s_2, \dots, s_n$, and those of $Q(G)$ by $q_1, q_2, \dots, q_n$. The Laplacian energy and the signless Laplacian energy of G, denoted by $\mathcal{L}E(G)$ and $QE(G)$, respectively, are given by

$$\mathcal{L}E(G) = \sum_{i=1}^{n} \left| s_i - \frac{2m}{n} \right|,\ QE(G) = \sum_{i=1}^{n} \left| q_i - \frac{2m}{n} \right|. \tag{4}$$

Several auxiliary results are used in the spectral analysis.

Lemma 1 [10] A graph G exhibits identical spectral for $\mathcal{L}(G)$ and $Q(G)$ if and only if it is bipartite.

Lemma 2 [8] Let A_1, A_2, A_3, A_4 be matrices of size $p \times p, p \times q, q \times p$, and $q \times q$, respectively. If A_1 or A_4 are invertible, then the determinant of the block matrix can be expressed as

$$\det \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \det(A) \det(A_4 - A_3 A_1^{-1} A_2) = \det(A_4) \det(A_1 - A_2 A_4^{-1} A_3).$$

Lemma 3 [8] Let M and N be block matrices of size $r \times r$, and consider

$$Y = \begin{bmatrix} M & N & N & \cdots & N & N \\ N & M & N & \cdots & N & N \\ N & N & M & \cdots & N & N \\ \vdots & \cdots & \cdots & \ddots & \ddots & \vdots \\ N & N & N & \cdots & M & N \\ N & N & N & \cdots & N & M \end{bmatrix}_{k \times k},$$

then, the determinant of Y can be expressed as

$$\det(Y) = (\det(M - N))^{k-1} (\det(M + (k-1)N)).$$

The main results of this study are obtained through a structured spectral analysis. First, the vertex partition of the lintang graph L_n induces a natural block structure in its associated matrices. This structure allows the adjacency, Laplacian, and signless Laplacian matrices to be expressed in block form. By applying block determinant techniques (Lemma 2 and Lemma 3), the characteristic polynomials are systematically reduced, leading to explicit expressions for the eigenvalues.

Next, the same analytical framework is extended to the line graph $L(L_n)$. In this transformation, each edge of L_n becomes a vertex in $L(L_n)$, resulting in a modified adjacency structure with a different but still regular block pattern. This regularity allows the eigenvalues of the associated matrices of $L(L_n)$ to be derived using analogous block matrix techniques.

Finally, the obtained eigenvalues are substituted into the definitions of graph energy, Laplacian energy, and signless Laplacian energy to derive explicit formulas. This procedure establishes a systematic connection between the structural properties of the lintang graph and the spectral energy invariants of its line graph.

To support the validity of the derived formulas, a direct computation is carried out for small values of n, such as $n = 3$. This step serves as an illustrative consistency check by comparing explicitly computed eigenvalues with the corresponding theoretical results. It is not intended as a large-scale computational analysis, but rather as a supporting step to confirm the correctness of the analytical derivations.

3. RESULT AND ANALYSIS

This section presents the main results on the spectral energies of the lintang graph L_n and its corresponding line graph $L(L_n)$. The analysis follows the algebraic framework described in Section 2, where the structural properties of the graph are translated into matrix representations and examined through spectral techniques. In particular, the adjacency matrix $A(G)$, Laplacian matrix $\mathcal{L}(G)$, and signless Laplacian matrix $Q(G)$ are constructed explicitly, and their eigenvalues are derived using block determinant methods as stated in Lemma 2 and Lemma 3.

To improve clarity, the presentation is divided into two subsections. The first focuses on the lintang graph L_n as the fundamental structure, while the second examines its line graph $L(L_n)$. This organization enables a direct comparison of how spectral energies evolve under graph transformation. In addition to the algebraic derivations, illustrative direct computations for small values of n are included to confirm the correctness of the obtained formulas. These computations serve as consistency checks rather than a large-scale computational experiment.

a. Lintang Graph

The adjacency matrix and the degree matrix of the lintang graph L_n are presented as follows. The adjacency matrix and the degree matrix of L_n is given by

$$A(L_n) = \begin{bmatrix} 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(n+2)\times(n+2)}, D(L_n) = \begin{bmatrix} n & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2 \end{bmatrix}_{(n+2)\times(n+2)}$$

From these matrices, the Laplacian matrix $\mathcal{L}(L_n)$ and the signless Laplacian matrix $Q(L_n)$ of the lintang graph L_n can be derived as

$$\mathcal{L}(L_n)=\begin{bmatrix} n & 0 & -1 & -1 & \cdots & -1 & -1\\ 0 & n & -1 & -1 & \cdots & -1 & -1\\ -1 & -1 & 2 & 0 & \cdots & 0 & 0\\ -1 & -1 & 0 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -1 & -1 & 0 & 0 & \cdots & 2 & 0\\ -1 & -1 & 0 & 0 & \cdots & 0 & 2\end{bmatrix}_{(n+2)\times(n+2)}, Q(L_n)=\begin{bmatrix} n & 0 & 1 & 1 & \cdots & 1 & 1\\ 0 & n & 1 & 1 & \cdots & 1 & 1\\ 1 & 1 & 2 & 0 & \cdots & 0 & 0\\ 1 & 1 & 0 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & 0 & 0 & \cdots & 2 & 0\\ 1 & 1 & 0 & 0 & \cdots & 0 & 2\end{bmatrix}_{(n+2)\times(n+2)}$$

Theorem 1. Let L_n be the lintang graph of order $n+2$ and size n. Then the following hold.
a) $E(L_n)=2\sqrt{2n}$;
b) $\mathcal{L}E(L_n)=QE(L_n)=\begin{cases}\frac{10}{3}, & n=1,\\ \frac{4(n^2-n+2)}{n+2}, & n\geq 2.\end{cases};$

Proof.
a) To determine the graph energy of L_n, we compute the characteristic polynomial of the adjacency matrix $A(L_n)$. Consider the matrix $\gamma I-A(L_n)$ and it's partition in block form

$$\gamma I-A(L_n)=\begin{bmatrix} \gamma & 0 & -1 & -1 & \cdots & -1 & -1\\ 0 & \gamma & -1 & -1 & \cdots & -1 & -1\\ -1 & -1 & \gamma & 0 & \cdots & 0 & 0\\ -1 & -1 & 0 & \gamma & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -1 & -1 & 0 & 0 & \cdots & \gamma & 0\\ -1 & -1 & 0 & 0 & \cdots & 0 & \gamma\end{bmatrix}_{(n+2)\times(n+2)} = \begin{bmatrix} A & B\\ B^T & C\end{bmatrix},$$

where

$$A=\begin{bmatrix}\gamma & 0\\ 0 & \gamma\end{bmatrix}, B=\begin{bmatrix}-1 & -1 & \cdots & -1\\ -1 & -1 & \cdots & -1\end{bmatrix}_{2\times n}, C=\begin{bmatrix}\gamma & 0 & \cdots & 0\\ 0 & \gamma & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \gamma\end{bmatrix}_{n\times n}$$

Using the determinant formula stated in Lemma 2, together with straightforward algebraic simplification, we obtain

$$\begin{aligned}\det(\gamma I-A(L_n)) &= \det(C).\det(A-BC^{-1}B^T)\\ &= \gamma^n(\gamma^2-2n)\end{aligned}$$

Hence, the spectrum of L_n is

$$Spec_A(L_n)=\begin{bmatrix}0 & \sqrt{2n} & -\sqrt{2n}\\ n & 1 & 1\end{bmatrix}$$

Hence using Definition 4, only the two nonzero eigenvalues contribute:

$$E(L_n)=|\sqrt{2n}|+|-\sqrt{2n}|=2\sqrt{2n}.$$

b) Consider the Laplacian matrix of L_n and form the matrix $sI-\mathcal{L}(L_n)$ are given:

$$sI-\mathcal{L}(L_n)=\begin{bmatrix} s-n & 0 & 1 & 1 & \cdots & 1 & 1\\ 0 & s-n & 1 & 1 & \cdots & 1 & 1\\ 1 & 1 & s-2 & 0 & \cdots & 0 & 0\\ 1 & 1 & 0 & s-2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 1 & 0 & 0 & \cdots & s-2 & 0\\ 1 & 1 & 0 & 0 & \cdots & 0 & s-2\end{bmatrix}_{(n+2)\times(n+2)} = \begin{bmatrix} A & B\\ B^T & C\end{bmatrix}$$

where, $A=\begin{bmatrix}s-n & 0\\ 0 & s-n\end{bmatrix}, B=\begin{bmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\end{bmatrix}_{2\times n}, C=\begin{bmatrix}s-2 & 0 & \cdots & 0\\ 0 & s-2 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & s-2\end{bmatrix}_{n\times n}$.

Then $\det(A) = (s-n)^2$ and $B^T A^{-1} B = \frac{2}{s-n} J_{n\times n}$ where $J_{n\times n}$ denotes the $n \times n$ all-ones matrix. Hence $C - B^T A^{-1} B = (s-2) I_n - \frac{2}{s-n} J_{n\times n}$. Because $J_{n\times n}$ has one eigenvalue n and $n-1$ eigenvalues 0, the determinant of $C - B^T A^{-1} B$ can be computed as

$$\det(C - B^T A^{-1} B) = \left(s - 2 - \frac{2n}{s-n}\right)(s-2)^{n-1} = \frac{(s-2)(s-n)-2n}{s-n} . (s-2)^{n-1}.$$

Applying determinant formula (with A nonsingular) in Lemma 2, yields

$$\begin{aligned} \det(sI - \mathcal{L}(L_n)) &= \det(A) . \det(C - B^T A^{-1} B) \\ &= (s-n)^2 \frac{(s-2)(s-n)-2n}{s-n} . (s-2)^{n-1} \\ &= s(s-2)^{n-1}(s-n)\big(s-(n+2)\big). \end{aligned}$$

Therefore the Laplacian spectrum of L_n is

$$Spec_{\mathcal{L}}(L_n) = \begin{bmatrix} 0 & 2 & n & n+2 \\ 1 & n-1 & 1 & 1 \end{bmatrix}.$$

Next, the Laplacian energy of the lintang graph L_n is given by:

$$\mathcal{L}E(L_n) = |0 - \bar{d}| + (n-1)|2 - \bar{d}| + |n - \bar{d}| + |(n+2) - \bar{d}|,$$

where $\bar{d}$ is the average degree:

$$\bar{d} = \frac{2|E|}{|V|} = \frac{4n}{n+2}.$$

Hence,

$$\mathcal{L}E(L_n) = \begin{cases} \frac{10}{3}, & n = 1, \\ \frac{4(n^2-n+2)}{n+2}, & n \geq 2. \end{cases}.$$

Since the lintang graph L_n is bipartite, it follows from Lemma 1 that

$$QE(L_n) = \mathcal{L}E(L_n) = \begin{cases} \frac{10}{3}, & n = 1, \\ \frac{4(n^2-n+2)}{n+2}, & n \geq 2. \end{cases}. \blacksquare$$

To verify the correctness of the obtained formulas, consider the case $n = 3$ (see Figure 1). In this case, the lintang graph L_3 has order 5 and size 6. The adjacency matrix, Laplacian matrix, and signless Laplacian matrix of L_3 are given by

$$A(L_3) = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \mathcal{L}(L_3) = \begin{bmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}. Q(L_3) = \begin{bmatrix} 3 & 0 & 1 & 1 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{bmatrix}.$$

Direct numerical computation of the eigenvalues of $A(L_3)$ produces the spectrum

$$Spec_A(L_3) = \begin{bmatrix} 0 & \sqrt{2(3)} & -\sqrt{2(3)} \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{6} & -\sqrt{6} \\ 3 & 1 & 1 \end{bmatrix}$$

Taking the sum of the absolute values confirms

$$E(L_3) = |0| + |\sqrt{6}| + |-\sqrt{6}| = 2\sqrt{6},$$

Substituting $n = 3$ into the derived formula yields

$$E(L_3) = 2\sqrt{2(3)} = 2\sqrt{6},$$

which agrees with the theoretical formula. A similar substitution for the Laplacian energy and signless Laplacian energy gives

$$\bar{d} = \frac{2|E|}{|V|} = \frac{2(6)}{5} = \frac{12}{5} = 2.4$$

$$Spec_{\mathcal{L}}(L_3) = \begin{bmatrix} 0 & 2 & 3 & 3+2 \\ 1 & 3-1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & 5 \\ 1 & 2 & 1 & 1 \end{bmatrix}$$

$$\mathcal{L}E(3) = |0 - \bar{d}| + 2|2 - \bar{d}| + |3 - \bar{d}| + |5 - \bar{d}| = |0 - 2.4| + 2|2 - 2.4| + |3 - 2.4| + |5 - 2.4| = 6{,}4$$

$$\mathcal{L}E(L_3) = QE(L_3) = \frac{4((3)^2 - 3 + 2)}{3+2} = \frac{4(9 - 3 + 2)}{5} = 6.4,$$

which agrees with the theoretical formula.

The obtained results show that the energy of the lintang graph grows as

$$E(L_n) = 2\sqrt{2n},$$

indicating sublinear growth with respect to the number of vertices. In contrast, both Laplacian energy and signless Laplacian energy grow linearly with respect to n, meaning that their values increase in direct proportion to the size of the graph. This difference reflects the heterogeneous structure of L_n, where two central vertices are connected to a set of peripheral vertices, resulting in only two nonzero adjacency eigenvalues.

b. Line Graph of Lintang Graph

From Definition 2, the adjacency matrix and degree matrix of $L(L_n)$ is obtained as

$$A(L(L_n)) = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & \cdots & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \end{bmatrix}_{(2n)\times(2n)}, D(L(L_n)) = \begin{bmatrix} n & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & n & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & n & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & n & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & n \end{bmatrix}_{(2n)\times(2n)}$$

Hence, the Laplacian matrix of the line graph is

$$\mathcal{L}(L(L_n)) = \begin{bmatrix} n & -1 & -1 & 0 & -1 & \cdots & -1 & 0 \\ -1 & n & 0 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & n & -1 & -1 & \cdots & -1 & 0 \\ 0 & -1 & -1 & n & 0 & \cdots & 0 & -1 \\ -1 & 0 & -1 & 0 & n & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & -1 & 0 & -1 & \cdots & n & -1 \\ 0 & -1 & 0 & -1 & 0 & \cdots & -1 & n \end{bmatrix}_{(n\times 2)\times(n\times 2)}.$$

Similarly, the signless Laplacian matrix is given by

$$Q(L(L_n)) = \begin{bmatrix} n & 1 & 1 & 0 & 1 & \cdots & 1 & 0 \\ 1 & n & 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & n & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & n & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & n & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 1 & 0 & 1 & \cdots & n & 1 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 1 & n \end{bmatrix}_{(n\times 2)\times(n\times 2)}.$$

Theorem 2. Let $L(L_n)$ be the line graph of the lintang graph. Then the graph energy of $L(L_n)$ is

$$E\big(L(L_n)\big) = \mathcal{L}E\big(L(L_n)\big) = QE\big(L(L_n)\big) = 4n - 4.$$

Proof.

To determine the eigenvalues of the adjacency matrix, consider the characteristic matrix

$$\gamma I - A\big(L(L_n)\big) = \begin{bmatrix} \gamma & -1 & -1 & 0 & -1 & \cdots & -1 & 0 \\ -1 & \gamma & 0 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & \gamma & -1 & -1 & \cdots & -1 & 0 \\ 0 & -1 & -1 & \gamma & 0 & \cdots & 0 & -1 \\ -1 & 0 & -1 & 0 & \gamma & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & -1 & 0 & -1 & \cdots & \gamma & -1 \\ 0 & -1 & 0 & -1 & 0 & \cdots & -1 & \gamma \end{bmatrix}_{(2n)\times(2n)}.$$

This matrix can be expressed in block form as

$$\gamma I - A\big(L(L_n)\big) = \begin{bmatrix} M & N & N & \cdots & N & N \\ N & M & N & \cdots & N & N \\ N & N & M & \cdots & N & N \\ \vdots & \cdots & \cdots & \ddots & \ddots & \vdots \\ N & N & N & \cdots & M & N \\ N & N & N & \cdots & N & M \end{bmatrix}_{n\times n}$$

with $M = \begin{bmatrix} \gamma & -1 \\ -1 & \gamma \end{bmatrix}, N = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

From direct computation,

$$M-N=\begin{bmatrix}\gamma+1 & -1\\ -1 & \gamma+1\end{bmatrix}$$

$$M+(n-1)N=\begin{bmatrix}\gamma+(n-1)(-1) & -1+(n-1)0\\ -1+(n-1)0 & \gamma+(n-1)(-1)\end{bmatrix}=\begin{bmatrix}\gamma-n+1 & -1\\ -1 & \gamma-n+1\end{bmatrix}.$$

Thus, using block determinant formulas in Lemma 3:

$$\begin{aligned}\det\left(\gamma I-A\big(L(L_n)\big)\right) &= (\det(M-N))^{n-1}\det(M+(n-1)N)\\ &= \left(\det\begin{bmatrix}\gamma+1 & -1\\ -1 & \gamma+1\end{bmatrix}\right)^{n-1}\det\begin{bmatrix}\gamma-n+1 & -1\\ -1 & \gamma-n+1\end{bmatrix}\\ &= ((\gamma+1)^2-1)^{n-1}((\gamma-n+1)^2-1)\\ &= (\gamma^2+2\gamma)^{n-1}(\gamma^2-2\gamma n+n^2+2\gamma-2n)\\ &= \gamma^{n-1}(\gamma+2)^{n-1}(\gamma-n+2)(\gamma-n).\end{aligned}$$

Hence, the characteristic polynomial is

$$p(L(L_n),\gamma)=\gamma^{n-1}(\gamma+2)^{n-1}(\gamma-n+2)(\gamma-n).$$

The adjacency spectrum is

$$Spec_A\,(L(L_n))=\begin{bmatrix}0 & -2 & n-2 & n\\ n-1 & n-1 & 1 & 1\end{bmatrix}.$$

Hence, the graph energy follows as

$$\begin{aligned}E(L(L_n)) &= (n-1)|0|+(n-1)|-2|+|n-2|+|n|\\ &= 2(n-1)+(n-2)+n\\ &= 4n-4.\end{aligned}$$

Using the same block decomposition for $sI-\mathcal{L}(L(L_n))$, we obtain the characteristic polynomial

$$p(L(L_n),s)=s(s-2)(s-n)^{(n-1)}(s-n-2)^{(n-1)},$$

which yields the Laplacian spectrum

$$Spec_{\mathcal{L}}\,(L(L_n))=\begin{bmatrix}0 & 2 & n & n+2\\ 1 & 1 & n-1 & n-1\end{bmatrix}.$$

Given that the graph has order $2n$ and size n^2, and that the average degree is $\bar{d}=\frac{2|E|}{|V|}=n$, the Laplacian energy can be computed as follows

$$\begin{aligned}\mathcal{L}E(L(L_n)) &= |0-n|+|2-n|+(n-1)|n-n|+(n-1)|n+2-n|\\ &= n+n-2+2n-2\\ &= 4n-4.\end{aligned}$$

A similar procedure can be applied to the signless Laplacian matrix of the line graph $L(L_n)$, yielding the characteristic polynomial

$$p(L(L_n),q)=(q-(n-2))^{n-1}(q-n)^{n-1}(q-(2n-2))(q-2n),$$

with the corresponding spectrum

$$Spec_Q(L(L_n))=\begin{bmatrix}n-2 & n & 2n-2 & 2n\\ n-1 & n-1 & 1 & 1\end{bmatrix}.$$

Using the same average degree $\bar{d}=n$ as in the Laplacian energy calculation of the line graph of L_n, the signless Laplacian energy is computed as

$$\begin{aligned}QE(L(L_n)) &= (n-1)|n-2-n|+(n-1)|n-n|+|2n-2-n|+|2n-n|\\ &= 2(n-1)+0+(n-2)+n\\ &= 4n-4.\ \blacksquare\end{aligned}$$

The coincidence of the three energy measures for the line graph $L(L_n)$ follows from the regularity of its structure and the resulting linear relations among the corresponding spectra. Since $L(L_n)$ is $(n+2)$ −regular, the adjacency, Laplacian, and signless Laplacian matrices satisfy

$$\mathcal{L}=(n+2)I-A,\ Q=(n+2)I+A.$$

Because these matrices are affine transformations of the same adjacency spectrum, all eigenvalue deviations used in their energy definitions ultimately depend only on the magnitudes of the adjacency eigenvalues. In the Laplacian-based energies, the central term is the average degree, which for any simple graph is

$$\bar{d}=\frac{2|E|}{|V|},$$

and this equals the regularity parameter of $L(L_n)$.

More precisely, letting $\{\gamma_i\}$ denote the adjacency eigenvalues of $L(L_n)$, the Laplacian and signless Laplacian eigenvalues satisfy

$$s_i=\frac{2|E|}{|V|}-\gamma_i,\ q_i=\frac{2|E|}{|V|}+\gamma_i.$$

It follows that

$$\left|s_i - \frac{2|E|}{|V|}\right| = |\gamma_i| = \left|q_i - \frac{2|E|}{|V|}\right|$$

so the ordinary energy, the Laplacian energy, and the signless Laplacian energy reduce to the same absolute spectral values. Therefore, all three energies of $L(L_n)$ are identical.

To further confirm the validity of the obtained formula, consider again the case $n = 3$ (see Figure 2). The line graph $L(L_3)$ contains 6 vertices and 9 edges. The adjacency matrix, Laplacian matrix, and signless Laplacian matrix of $L(L_3)$ are given by

$$A(L(L_3)) = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \mathcal{L}(L(L_3)) = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & 0 & -1 & 0 & -1 \\ -1 & 0 & 3 & -1 & -1 & 0 \\ 0 & -1 & -1 & 3 & 0 & -1 \\ -1 & 0 & -1 & 0 & 3 & -1 \\ 0 & -1 & 0 & -1 & -1 & 3 \end{bmatrix}. Q(L(L_3)) = \begin{bmatrix} 3 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 1 & 0 & 1 \\ 1 & 0 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 3 & 0 & 1 \\ 1 & 0 & 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 1 & 1 & 3 \end{bmatrix}.$$

Direct numerical computation of the adjacency eigenvalues yields

$$Spec_A\,(L(L_3)) = \begin{bmatrix} 0 & -2 & 3-2 & 3 \\ 3-1 & 3-1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

The sum of their absolute values equals 8.

Substituting $n = 3$ into the formula gives $E(L(L_3)) = 4(3) - 4 = 8$, which agrees with the theoretical formula. A similar substitution for the Laplacian energy and signless Laplacian energy gives

$$\bar{d} = \frac{2|E|}{|V|} = \frac{2(9)}{6} = 3$$

$$Spec_{\mathcal{L}}\,(L(L_3)) = \begin{bmatrix} 0 & 2 & 3 & 3+2 \\ 1 & 1 & 3-1 & 3-1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & 5 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

$$\mathcal{L}E(L_3) = |0-3| + |2-3| + 2|3-3| + 2|5-3| = 8.$$

$$\mathcal{L}E(L_3) = 4n - 4 = 4(3) - 4 = 8.$$

$$Spec_Q(L(L_3)) = \begin{bmatrix} 3-2 & 3 & 2(3)-2 & 2(3) \\ 3-1 & 3-1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 & 6 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

$$QE(L(L_3)) = 2|1-3| + 2|3-3| + |4-3| + |6-3| = 8.$$

$$QE(L(L_3)) = 4(3) - 4 = 8,$$

which agrees with the theoretical formula.

In contrast to the original graph, the energy of the line graph grows linearly as

$$E(L(L_n)) = 4n - 4.$$

Moreover, the equality

$$E(L(L_n)) = \mathcal{L}E(L(L_n)) = QE(L(L_n))$$

holds due to the regular structure of the line graph, where all vertices have equal degree. This regularity induces affine relationships between the adjacency, Laplacian, and signless Laplacian spectra, causing all three energy measures to coincide.

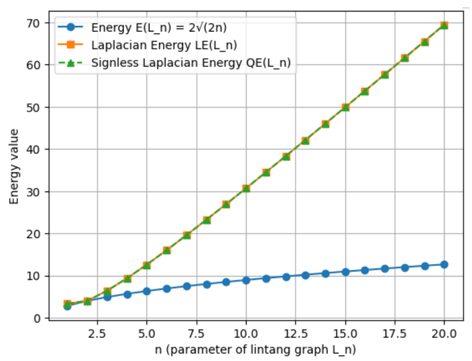

Figure 3. Energy vs Laplacian Energy vs Signless Laplacian Energy of Lintang Graph

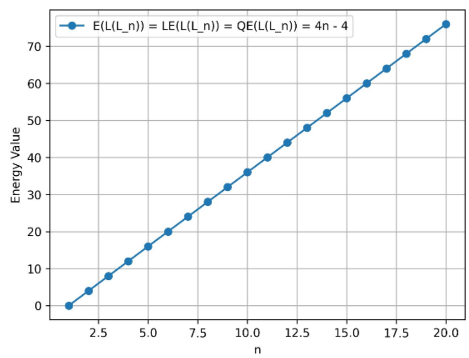

Figure 4. Energy vs Laplacian Energy vs Signless Laplacian Energy of Line Graph of Lintang Graph

Figures 3 and 4 illustrate the behavior of the energy, Laplacian energy, and signless Laplacian energy as functions of n. Figure 3 shows that for the lintang graph L_n, the adjacency energy increases more slowly compared to the Laplacian-based energies, reflecting its sublinear growth. In contrast, Figure 4 demonstrates that all three energy measures of the line graph $L(L_n)$ coincide and increase linearly with respect to n. These plots visually confirm the analytical results and highlight the significant impact of the line graph transformation on spectral energy behavior.

The obtained results can be interpreted within the broader framework of spectral graph theory. Previous studies, such as Energy of line graphs [10], have shown that the energy of line graphs is closely related to the structural properties of the original graph. Furthermore, Relation between signless Laplacian energy and graph energy [11] and Signless Laplacian energy of a graph and energy of a line graph [12] established general relationships between adjacency-based and Laplacian-based energies. The present findings are consistent with these results and provide a concrete example where all three energy measures coincide.

From a structural perspective, the difference in growth behavior between L_n and $L(L_n)$ is particularly significant. The sublinear growth of $E(L_n)$ reflects the concentration of connectivity around two central vertices, whereas the linear growth of $E(L(L_n))$ indicates that the line graph transformation redistributes adjacency relations more uniformly across vertices. This transformation effectively converts a two-hub structure into a regular graph, leading to a more evenly distributed spectral profile.

These results also have potential implications in applications. In network analysis, graphs with two dominant vertices can serve as simplified models of communication systems with central controllers, while their line graphs represent interactions between communication links. In theoretical chemistry, similar structures may arise in molecular graphs with two principal atoms connected to multiple substituents, where spectral energy is related to molecular stability.

Finally, this study opens several directions for further research. One possible extension is to investigate joins of independent sets involving more than two central vertices, leading to multi-hub graph structures. Another direction is to explore how other graph transformations, such as subdivision graphs or total graphs, influence spectral energy measures. These generalizations would contribute to a broader understanding of how structural modifications affect spectral invariants in complex networks.

4. CONCLUSION

This study establishes exact formulas for the energy, Laplacian energy, and signless Laplacian energy of the lintang graph L_n and its line graph $L(L_n)$, providing explicit spectral characterizations for a structured two-hub graph class that has not been previously reported in this form. The results demonstrate that $E(L_n) = 2\sqrt{2n}$, while $\mathcal{L}E(L_n) = QE(L_n)$ are expressed as rational functions of n; moreover, the transformation to the line graph results in the relation $E(L(L_n)) = \mathcal{L}E(L(L_n)) = QE(L(L_n)) = 4n - 4$. Taken together, these formulas show that the heterogeneous two-hub structure of L_n produces sublinear growth in adjacency energy, whereas the line graph transformation reorganizes edge interactions into a more uniform configuration that induces linear growth and spectral equivalence across the three energy measures. These findings provide a clear structural–spectral interpretation that can be used to benchmark energy behavior in other non-regular graph families and to validate newly derived spectral formulas. In addition, the explicit expressions offer practical utility in chemical graph theory, where energy-related indices are used to approximate molecular stability, as well as in network design, where controlling spectral properties is essential for optimizing connectivity patterns. More broadly, the results highlight how structural transformations systematically reshape spectral invariants, offering a concrete reference for analyzing similar graph constructions. This framework can be extended to more general graph families, including multi-hub configurations and alternative graph transformations, to further investigate how structural modifications influence spectral energy measures in complex networks.

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5. REFERENCES

[1] M. A. A. De Freitas, A. S. Bonifácio, M. Robbiano, and B. San Martín, “On matrices associated to directed graphs and applications,” *Linear algebra and its applications*, vol. 442, pp. 156–164, Feb. 2014, doi: 10.1016/j.laa.2013.07.005.

[2] Q. Li and L. Tang, “Infinite numbers of infinite classes l-borderenergetic graphs,” *Match - Communications in mathematical and in computer chemistry*, vol. 90, no. 3, pp. 729–742, Jul. 2023, doi: 10.46793/match.90-3.729L.

[3] K. R. Saoub, *Graph theory: an introduction to proofs, algorithms, and applications*, 1st ed. in Textbooks in mathematics. Boca Raton: CRC Press, 2021. doi: 10.1201/9781138361416.

[4] V. Zverovich, *Modern Applications of Graph Theory*, 1st ed. Oxford: Oxford university press, 2021. doi: 10.1093/oso/9780198856740.001.0001.

[5] A. Alexander, F. Fran, and N. Kusumastuti, “Complexity of Graphs with Wheel Graph and Fan Graph as their Blocks,” *JTAM (Jurnal teori dan aplikasi matematika)*, vol. 9, no. 4, p. Art. no. 1218, Oct. 2025, doi: 10.31764/jtam.v9i4.32776.

[6] F. Fran, Alexander, Yundari, P. Romanda, and E. Febyolga, “The Complexity of Octopus Graph, Friendship Graph, and Snail Graph,” *EduMatSains : Jurnal pendidikan, matematika dan sains*, vol. 9, no. 1, pp. 84–101, Jul. 2024, doi: 10.33541/edumatsains.v9i1.6042.

[7] C. Hanssen, F. Fran, and Y. Yundari, “The Complexity of Pencil Graph and Line Pencil Graph,” *PYTHAGORAS Jurnal pendidikan matematika*, vol. 19, no. 2, pp. 115–125, Nov. 2024, doi: 10.21831/pythagoras.v19i2.77747.

[8] M. R. Zeen El Deen, W. A. Aboamer, and H. M. El-Sherbiny, “The Complexity of the Super Subdivision of Cycle-Related Graphs Using Block Matrices,” *Computation*, vol. 11, no. 8, p. 162, Aug. 2023, doi: 10.3390/computation11080162.

[9] P. Kornelia, F. Fran, and E. Noviani, “Bilangan B-Kromatik Pada Graf Origami, Graf Lintang, Dan Graf Tadpole,” *Bimaster : Buletin ilmiah matematika, statistika dan terapannya*, vol. 8, no. 4, Oct. 2019, doi: 10.26418/bbimst.v8i4.36551.

[10] I. Gutman, M. Robbiano, E. A. Martins, D. M. Cardoso, L. Medina, and O. Rojo, “Energy of line graphs,” *Linear algebra and its applications*, vol. 433, no. 7, pp. 1312–1323, Dec. 2010, doi: 10.1016/j.laa.2010.05.009.

[11] K. Ch. Das and S. A. Mojallal, “Relation between signless Laplacian energy, energy of graph and its line graph,” *Linear algebra and its applications*, vol. 493, pp. 91–107, Mar. 2016, doi: 10.1016/j.laa.2015.12.006.

[12] H. A. Ganie, B. A. Chat, and S. Pirzada, “Signless Laplacian energy of a graph and energy of a line graph,” *Linear algebra and its applications*, vol. 544, pp. 306–324, May 2018, doi: 10.1016/j.laa.2018.01.021.

[13] I. Gutman and B. Zhou, “Laplacian energy of a graph,” *Linear algebra and its applications*, vol. 414, no. 1, pp. 29–37, Apr. 2006, doi: 10.1016/j.laa.2005.09.008.

[14] I. Gutman and B. Furtula, “Survey of Graph Energies,” *Mathematics interdisciplinary research*, vol. 2, no. 2, Sep. 2017, doi: 10.22052/mir.2017.81507.1057.

[15] A. Raza, M. Munir, M. Hussain, and F. Tasgera, “A Spectrum-Based Approach to Network Analysis Utilizing Laplacian and Signless Laplacian Spectra to Torus Networks,” *IEEE Access*, vol. 12, pp. 52016–52029, 2024, doi: 10.1109/ACCESS.2024.3384300.

[16] S. Srinivasan and V. Shanmugavel, “Comparative study on different types of energies,” *Indian journal of science and technology*, vol. 17, no. 28, pp. 2954–2959, Jul. 2024, doi: 10.17485/IJST/v17i28.1447.

[17] Y. Chen, S. Liu, and Z. Wang, “Signless Laplacian energy and spectral radius of a graph,” *Discrete applied mathematics*, vol. 378, pp. 216–225, Jan. 2026, doi: 10.1016/j.dam.2025.07.015.

[18] P. Bora and M. Rahman, “On The Spectral Characteristics Of Signless Laplacian Matrix,” *South east asian journal of mathematics and mathematical sciences*, vol. 21, no. 2, pp. 43–52, Aug. 2025, doi: 10.56827/SEAJMMS.2025.2102.3.

[19] G. H. FathTabar, “On the Signless Laplacian Eigenvalues and Optimum SLE of Graph,” *Fuzzy optimization and modeling journal*, vol. 3, no. 2, pp. 40–45, Jun. 2022, doi: 10.30495/fomj.2022.1950857.1059.

[20] S. Banerjee, “The Signless Laplacian Spread of Power Graphs of Finite Groups,” *Ikonion journal of mathematics*, vol. 7, no. 2, pp. 24–35, Dec. 2025, doi: 10.54286/ikjm.1658465.

[21] A. Alhevaz, M. Baghipur, S. Pirzada, and Y. Shang, “Some bounds for distance signless Laplacian energy-like invariant of networks,” *Carpathian mathematical publications*, vol. 17, no. 1, pp. 255–276, Jun. 2025, doi: 10.15330/cmp.17.1.255-276.

[22] U. Preetha P, M. Suresh, and E. Bonyah, “On the spectrum, energy and Laplacian energy of graphs with self-loops,” *Heliyon*, vol. 9, no. 7, p. e17001, Jul. 2023, doi: 10.1016/j.heliyon.2023.e17001.

[23] A. Bilal and M. Mobeen Munir, “Modified Sombor Spectral Radii and Modified Sombor Energies of Splitting and Shadow Graphs,” *Scientific annals of computer science*, vol. 34, no. 2, pp. 89–112, 2024, doi: 10.47743/SACS.2024.2.89.

[24] A. Mohamed Atheeque and S. Sharief Basha, “Signless Laplacian energy aware decision making for electric car batteries based on intuitionistic fuzzy graphs,” *Science progress*, vol. 107, no. 4, p. 00368504241301813, Oct. 2024, doi: 10.1177/00368504241301813.

[25] R. Sarathy and J. R. Sankar, “Laplacian energy and color based energy for graphs associated with commutative rings,” *Journal of discrete mathematical sciences and cryptography*, vol. 27, no. 8, pp. 2431–2445, 2024, doi: 10.1080/09720529.2024.2338813.

[26] B. A. Rather, M. Aijaz, F. Ali, N. Mlaiki, and A. Ullah, “On distance signless Laplacian eigenvalues of zero divisor graph of commutative rings,” *AIMS Mathematics*, vol. 7, no. 7, pp. 12635–12649, 2022, doi: 10.3934/math.2022699.

[27] K. Sato and M. Suzuki, “The nearest graph Laplacian in Frobenius norm,” *Numerical linear algebra with applications*, vol. 31, no. 6, p. e2578, Dec. 2024, doi: 10.1002/nla.2578.

[28] N. A. Mohd Rizal, N. I. Alimon, M. Selvarajoo, and N. H. Sarmin, “Laplacian Spectrum and Laplacian Energy of the Zero Divisor Graph for Z_3alpha,” *Malaysian journal of fundamental and applied sciences*, vol. 21, no. 6, pp. 2926–2936, Dec. 2025, doi: 10.11113/mjfas.v21n6.4303.

[29] N. F. Yalçın, “On Laplacian Energy of r-Uniform Hypergraphs,” *Symmetry*, vol. 15, no. 2, p. 382, Feb. 2023, doi: 10.3390/sym15020382.

[30] N. Nurliantika, F. Fran, and Y. Yundari, “Determinants of Tridiagonal and Circulant Matrices Special Form by Chebyshev Polynomials,” *JTAM (Jurnal teori dan aplikasi matematika)*, vol. 9, no. 1, p. 190, Jan. 2025, doi: 10.31764/jtam.v9i1.27871.