Zero : Jurnal Sains, Matematika, dan Terapan
E-ISSN : 2580-5754; P-ISSN : 2580-569X
Volume 10, Number 1, 2026
DOI: 10.30829/zero.v10i1.28727
Page: 392-398

Exponent and Scrambling Index of Some Composite Graphs

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Article Info

Article history:

Accepted 15 May 2026

Keywords:

Composite Graphs;
Exponent;
Scrambling Index.

ABSTRACT

A connected graphs G is said to be primitive if there exists a positive integer k such that, for each pair of vertices u and v, there is a uv-walk of length k. The scrambling index of a primitive graph G, denoted $k(G)$, is the smallest positive integer k such tha,t for any two vertices u and v, there exists a vertex w with the property that both a uw-walk and a $vw-$walk of length k exist. In this study, we analyze the scrambling index and exponent of joint and corona products of vertex-disjoint graphs. We present methods for calculating the lower and upper bounds of the scrambling index for these composite graphs and discuss their primitivity. The results show that both the joint and corona product graphs are primitive, and we provide explicit formulas for their exponent and scrambling index. Conceptually, these results extend the theory of primitive graphs to composite graphs and clarifies the relationship between graph topology, walk length, exponent, and scrambling index.

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1. INTRODUCTION

For some positive integer m, a walk of length m connection vertices u and v is a sequence of edges of the form

$$\{u=v_0,v_1\},\{v_1,v_2\},\ldots,\{v_{m-1},v_m=v\}$$

or

$$u=v_0-v_1-v_2-\ldots-v_{m-1}-v_m=v$$

A walk W connecting u and v is either denoted by $uv-$walk W or simply W_{uv}. A walk of length m connneting u and v is denoted by $u\_^{m}\_v$. A $uv-$walk is open when ever $u \neq v$ and is closed otherwise. A path is a walk without repeated vertices except possibly the end vertices. A cycle is a closed path. A triangle is a cycle of length 3. For two vertices u and v, the distance between u and v, $d(u,v)$, is the length of the shortest path connecting u and v. The diameter of a connected graph G is $diam(G)=\{d(u,v)\}$.

A graph G is connected if, for each pair of vertices u and v, there exists a $uv-$walk. A connected graph is primitive if there exists a positive integer k such that, for each pair of vertices u and v, a $uv-$walk of length k exists. The smallest such positive integers k is called the exponent of G. The exponent of a graph G is denoted by $\exp(G)$. It is well known that a connected graph G is primitive if and only if G contains a cycle of odd length [1]

In 2009, Akelbek and Kirkland [2], [3] introduced a new parameter of primitive graph called scrambling index. The scrambling index of primitive graph G is the smallest positive integer k such that for each pair of vertices u and v in G there is a vertex w in G such that there are a $uw-$walk and a $vw-$walk of length k. Equivalently, the scrambling index of a primitive graph is the smallest positive integer k such that for each pair of vertices u and v in G there is a $uv-$walk of length $2k$. The scrambling index of a primitive graph G is denoted by $k(G)$.

Chen and Liu [4] obtained an upper bound for scrambling index of primitive graphs in term of the length of the smallest odd cycle in G. Their result showed that if G is primitive graph on n vertices with the length of the smallest odd cycle in G is s, then $k(G) \le (s-1)/2+(n-s)$.

For vertex-disjoint connected graphs G and H, we discuss the exponent and the scrambling index of the joint $G+H$ and of the corona product $G \circ H$. In section 2, we present a way in setting up a lower bound and an upper bound for the scrambling index. In section 3, we discuss the primitivity of the joint and corona product of vertex-disjoint connected graphs. Finally, in section 4, we present formulae for exponents and the scrambling index of $G+H$ and $G \circ H$, respectively.

2. BOUNDS ON EXPONENT AND SCRAMBLING INDEX

This study employeds a theoretical-analytical framework in graph theory to determine the exponent and scrambling index of the join $G+H$ and the corona product $G \circ H$. Throughout this paper, G and H are assumed to be simple, connected, and vertex-disjoint graphs. The derivation proceeds in four steps. First, we used the definitions of primitive graphs, exponent, and scrambling index as the conceptual basis. Second, we established a basic extension property of walks, which is then used to relate walk lengths to the exponent and local scrambling index. Third, we derived lower and upper bounds for these parameters. Finally, we applied these general results to the join and corona product to obtain explicit formulas for their exponent and scrambling index.

First we discussed a property of $uv-$walk necessary for our discussion.

Proposition 2.1 *Let G be a connected graph. Every $uw-$ walk of length m can be extended to a $uw-$ walk of length $m+2t$* for each integer $t \ge 1$.

Proof. Let

$$W_{uv} : u = v_0 - v_1 - v_2 - \ldots - v_{m-1} - v_m = v$$

be a $uv-$walk of length m. Then the walk that starts at u, moves to the vertex v along the walk W_{uv}, and finally moves t times around the cycle $v - v_{m-1} - v$ is a $uv-$ walk of length $m+2t$.

Let G be a primitive graph. For vertices u and v, we define $\exp(G:u,v)$ to be the smallest positive integer k such that there is a $uv-$walk of length t for all positive integers $t \ge k$. We defined the *local scrambling index* of u and v in G, $k_{u,v}(G)$, to be the smallest positive integer k such that there is an even $uv-$walk in G of length $2k$. As a direct consequence of Proposition 2.1, we have the following relationship.

Corollary 2.2 *Let G be a primitive graph. For any pair of vertices u and v, $k_{u,v}(G) \le k(G)$ and $\exp(G:u,v) \le \exp(G)$. Moreover,*

$$k(G) = \max_{u,v}\{k_{uv}(G)\} \text{ and } \exp(G) = \max_{u,v}\{\exp(G:u,v)\}$$

We now present a lower bound for the exponent and scrambling index of a primitive graph.

Lemma 2.3 *Let G be a primitive graph. Then* $\exp(G) \ge diam(G)$ and $k(G) \ge \lceil diam(G)/2 \rceil$.

Proof. Let u and v be vertices in G such that $diam(G) = d(u,v)$. It is clear that $\exp(G) \ge \exp(G:u,v) \ge diam(G)$. Since the shortest even $uv-$walk is either of length at least $d(u,v)$ or of length at least $d(u,v)+1$, then

$$k_{u,v}(G)\begin{cases} diam(G)+2, & \text{if } diam(G) \text{ is even} \\ (diam(G)+1)/2, & \text{if } diam(G) \text{ is odd} \end{cases}$$

Therefore $k_{uv}(G) \ge \lceil diam(G)/2 \rceil$. By Corollary 2.2, we now conclude that $k(G) \ge k_{uv}(G) \ge \lceil diam(G)/2 \rceil$

We now present an upper bound for the exponent and scrambling index of primitive graph.

Lemma 2.4 *Let G be a primitive graph and let k be a positive integer. If for each pair of vertices u* and *v in G, there is a $uv-$ walk of length $\ell \le k$ and $\ell \equiv k \bmod 2$*, then $\exp(G) \le k$.

Proof. We show that for each pair of vertices u and v, , there is a $uv-$walk of length *k*. Let W_{uv} be a $uv-$walk of length $\ell(W_{uv}) \le k$. Since $\ell(W_{uv}) \equiv k \bmod 2$, $k\ell(W_{uv}) = 2t$ for some positive integer *t*. Proposition 2.1 guarantees that W_{uv} can be extended to a $uv-$walk of length $k = \ell(W_{uv}) + 2t$. Thus $\exp(G) \le k$.

Lemma 2.5 *Let G be a primitive graph and let ℓ be an even positive integer. If for each pair of vertices u* and *v, in G, there is an even $uv-$ walk of length at most ℓ, then $k(G) \le \ell/2$.*

Proof. It suffices to show that for each pair of vertices u and v, there is a $uv-$walk of length ℓ. Let W_{uv} be an even $uv-$walk of length $\ell(W_{uv}) \le \ell$. Since $\ell - \ell(W_{uv}) = 2t$ for some nonnegative integer *t*, then by Proposition 2.1 the walk W_{uv} can be extended to a walk W'_{uv} of length exactly ℓ. This implies that for each pair of vertices u and v there is a vertex w such that there exists an $uw-$walk of length $\ell/2$ and $vw-$walk of length $\ell/2$. Hence $k(G) \le \ell/2$.

The following result, due to Chen and Liu [4], gives a relationship between the exponent and scrambling index of primitive graph. For the sake of completenesss we present the proof.

Theorem 2.6 [4] *Let G be a primitive graph with exponent* $\exp(G)$. Then $k(G) = \lceil \exp(G)/2 \rceil$.

Proof. Since the exponent of G is $\exp(G)$, then there are vertices u and v such that $\exp_{uv}(G) = \exp(G)$. If $\exp(G)$ is even, then $k_{uv}(G) = \exp(G)/2$. If $\exp(G)$ is odd, then $k_{uv}(G) = (\exp(G)+1)/2$. Therefore, $k(G) \ge k_{uv}(G) = \lceil \exp(G)/2 \rceil$.

Since the exponent of G is $\exp(G)$, then for each pair of vertices u and v there is an even $uv-$walk of length either $\exp(G)$ of $\exp(G)+1$. Hence Lemma 2.5 guarantees that $k(G) \le \lceil \exp(G)/2 \rceil$.

3. SOME COMPOSITE GRAPHS

In the next section, the general bounds established above are applied to two composite graph classes, namely the join and corona product of connected vertex-disjoint graphs.

Let G and H be simple vertex-disjoint graphs, that is $V(G) \cap V(H) = \varnothing$. $V(G) = \{g_1, g_2, \ldots, g_{|V(G)|}\}$ and $H(G) = \{h_1, h_2, \ldots, h_{|V(H)|}\}$. The joint of G and H_i, $G+H$, is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set

$$E(G+H) = E(G) \cup E(H) \cup \{\{g_i, h_j\} : g_i \in V(G), 1 \le i \le |V(G)| \text{ and } h_j \in V(H), 1 \le j \le |V(H)|\}.$$

An illustration of a joint product of G and H is given in Figure 1.

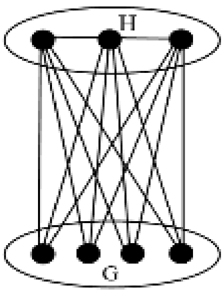

Figure 1. Joint G + H

The corona product of simple vertex-disjoint graphs G and H, $G \circ H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and by joining each certex of the *i*th copy of H to the *i*th vertex of G, for $1 \le i \le |V(G)|$ [5], [6]. For $i = 1, 2, \ldots, |V(G)|$, let H_i be the *i*th copy of H and $V(H_i) = \{h_{i1}, h_{i2}, \ldots, h_{i|V(H)|}\}$. Then $G \circ H$ is the graph with the vertex set

$$V(G \circ H) = V(G) \cup \left(\bigcup_{i=1}^{|V(G)|} V(H_i) \right)$$

and edge set

$$E(G \circ H) = E(G) \cup \left(\bigcup_{i=1}^{|V(G)|} E(H_i) \right) \cup \left\{ \{g_i, h_{ij}\} : g_i \in V(G), h_{ij} \in V(H_i), 1 \le i \le |V(G)|, 1 \le j \le |V(H)| \right\}$$

An illustration of the corona product $G \circ H$ is given in Figure 2.

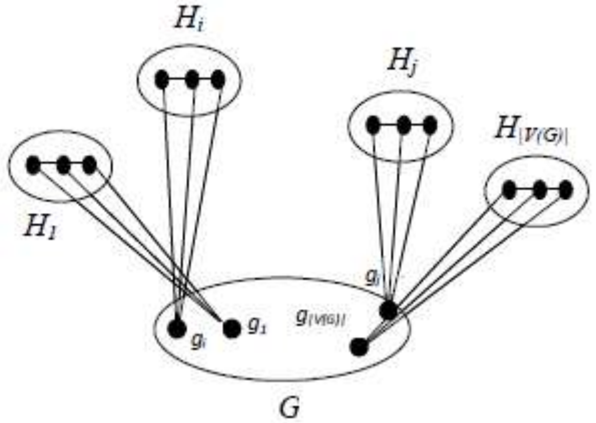

Figure 2. Corona Product $G \circ H$

Theorem 3.1 *Let G and H be vertex-disjoint connected graphs. Then the joint and corona product of G and H are primitive.*

Proof. We first show that the joint of G and H is primitive. Since G and H are connected and each vertex in G is adjacent to every vertex of H, $G+H$ is connected. Let u and v be two adjacent vertices in G. Since there is a vertex, say h, in H such that h is adjacent to both u and v, the joint $G+H$ contains a cycle of length 3. Hence, $G+H$ is primitive.

We now show that the corona product $G \circ H$ is primitive. Since G and H are connected and every vertex in each copy H_i of H, $1 \le i \le |V(G)|$, is adjacent to the vertex g_i of G, then $G \circ H$ is connected. Let h_{ij} and h_{ik} be two adjacent vertices in the i-th copy H_i of H, then the path $h_{ij} - h_{ik} - g_i - h_{ij}$ is a cycle of length 3 in $G \circ H$ is primitive.

4. RESULT

Theorem 4.1 *Let G and H be vertex-disjoint connected graphs. Then* $\exp(G+H) = 2$

Proof. Since there is no uu-walk of length 1, $\exp(G+H) \ge 2$. It remains to show that $\exp(G+H) \le 2$, i.e., for each pair of vertices u and v, there is a uv – walk of length 2. If $u = v$, it is clear that there is a uu-walk of length 2. We now assume that $u \ne v$. Suppose $u \in G$ and $v \in H$. Since G is connected, then there is a vertex $w \in G$ that adjacent to both u and v. Thus, there is a triangle that contains u and v, and hence there is a uv – walk of length 2. If both u and v are in G, then there is a vertex $w \in H$ such that the walk $u - w - v$ is a uv – walk of length 2. The same result applies when both u and v lie on H. Therefore, for each pair of vertices u and v, there is a uv – walk of length 2. Hence $\exp(G+H) \le 2$.

Theorem 4.1 shows that for two connected vertex-disjoint graphs G and H, the join $G+H$ has an exponent exactly equal to 2, that is $\exp(G+H) = 2$. This means that the join structure creates uniformly short walks between every pair of vertices, ensuring strong primitive connectivity. Hence, the join produces a highly efficient primitive graph in terms of walk synchronization.

As a direct consequence of Theorem 2.6 and Theorem 4.1 we have the following result.

Corollary 4.2 *Let G and H be vertex-disjoint connected graph. Then $k(G+H) = 1$.*

From the relationship between exponent and scrambling index, it follows that $k(G+H) = 1$. This confirms that the join is not only primitive, but also has a very fast scrambling process, since any two vertices can reach a common vertex after one appropriate walk step.

We now discuss the exponent of the corona product $G \circ H$.

Theorem 4.3 *Let* G *and* H *be vertex-disjoint connected graphs. Then* $\exp(G\circ H)=diam(G)+2$.

Proof. For $i=1,2,\ldots,|V(G)|$, let H_i be the i-th copy of H . Since every vertex $h_{ij}\in H_i$ is adjacent to a vertex in G and no edge connect the vertices in different copies H_i of H , the $diam(G\circ H)=diam(G)+2$. Thus, by Lemma 2.3, we have $\exp(G\circ H)\geq diam(G)+2$.

We next show that $k(G\circ H)\leq diam(G)+2$, i.e. For each pair of vertices u and v in $(G\circ H)$ there is a $uv-$walk W_{uv} of length $\ell(W_{uv})\equiv(diam(G)+2)$. If $d(u,v)\equiv(diam(G)+2)\bmod 2$, so we are done. Thus, we assume that $d(u,v)\not\equiv(diam(G)+2)\text{mod}2$.

Since for each vertex u lies on a cycle of length 2 and on a cycle of length 3, then there a closed uu-walk W_{uv} of length $\ell(W_{uv})\equiv(diam(G)+2)\bmod 2$. Therefore, we now assume that $u\neq v$. Suppose $u=g_i$ and $v=g_j$ for some $1\leq i,j\leq|V(G)|$, i.e. both u and v lie on G . Let $h_{i,j}$ and $h_{i,t}$ be two adjacent vertices in H_i . Since $d(u,v)\not\equiv(diam(G)+2)\text{mod}2$, then $d(u,v)<diam(G)$ and the $uv-$walk.

$$u=g_i-h_{i,j}-h_{i,t}-u=g_i\overset{d(g_i,g_j)}{\underset{--}{}}v=g_j$$

Is a $uv-$walk of length $\ell(W_{uv})\equiv(diam(G)+2)$ mod 2. If $u\in V(H_i)$ and $v\in V(H_j)$ for some $i\neq j$, then $u=h_{i,k}$ for some $1\leq j\leq|V(H)|$ and $v=h_{j,\ell}$ for some $1\leq\ell\leq|V(H)|$. Suppose $d(u,v)$ is obtained by the path

$$u-h_{i,k}-g_i\overset{d(g_i,g_j)}{\underset{--}{}}g_j-h_{j,\ell}=v$$

Then $d(g_i,g_j)$ $d(u,v)\not\equiv(diam(G)+2)\text{mod}2$. Since H is connected there is a vertex $h_{i,p}$ adjacent to $h_{i,k}$. This implies the $uv-$path.

$$P_{uv}:u=h_{i,k}-h_{i,p}-g_i\overset{d(g_i,g_j)}{\underset{--}{}}g_j-h_{j,\ell}=v$$

Is a $uv-$path such that $\ell(P_{uv})\equiv(diam(G)+2)\bmod 2$. Since $d(g_i,g_j)\leq diam(G)$, we have $\ell(P_{uv})\leq diam(G)+2$.

Suppose $u=h_{i,j}\in V(H_i)$ for some $1\leq i\leq|V(G)|$ and $v=g_k\in G$. Then $d(u,v)$ is obtained by a shortest $uv-$path of the form

$$u=h_{i,j}-g_i\overset{d(g_i,g_j)}{\underset{--}{}}g_k=v$$

Hance $d(g_i,g_k)\equiv(diam(G)+2)\bmod 2$ and since H is connected, there is a vertex $h_{i,t}\in H_i$ adjacent to $h_{i,j}$. Then the $uv-$walk

$$W_{uv}:u=h_{i,j}-h_{i,t}-g_i\overset{d(g_iv=g_k)}{\underset{--}{}}g_k=v$$

Is a $uv-$walk such that $\ell(W_{uv})\equiv(diam(G)+2)\bmod 2$. Since $d(g_i,g_k)\equiv(diam(G)+2)\bmod 2$, we have $\ell(W_{uv})\leq diam(G)+2$.

Therefore, for each pair of vertices u and v in $G\circ H$, there is a walk W_{uv} connecting u and v of length $\ell(W_{uv})\leq diam(G)+2$ and $\ell(W_{uv})\equiv(diam(G)+2)\bmod 2$. Lemma 2.5 guarantees that $\exp(G\circ H)\leq(diam(G)+2)$.

Theorem 4.3 establishes that the exponent of the corona product is $\exp(G\circ H)=diam(G)+2$. This formula shows that, unlike the join, the exponent of the corona product is not constant, but depends directly on the diameter of the graph G . Therefore, the larger the diameter of G , the longer the uniform walk length required, the more constrained the walk structure in the corona product.

Corollary 4.4 *Let* G *and* H *be a vertex-disjoint connected graph. Then* $k(G\circ H)=\lceil(diam(G)+2)/2\rceil$.

Proof. This is a direct consequence of Theorem 2.6 and Theorem 4.3

As a consequence, the scrambling index of the corona product is $k(G \circ H) = \lceil (diam(G)+2)/2 \rceil$. This result shows that the scrambling index behavior of the corona product is also controlled by the diameter of the base graph G. Hence, the diameter plays an essential role in determining the scrambling efficiency of the corona product.

5. CONCLUSION

This study determines explicit formulas for the exponent and scrambling index of two classes of composite graphs, namely the join $G+H$ and the corona product $G \circ H$, under the assumption that the G and H be vertex-disjoint connected graphs. The main results show that $\exp(G+H)=2$, $k(G+H)=1$ and $\exp(G \circ H) = diam(G)+2$, $k(G \circ H) = \lceil (diam(G)+2)/2 \rceil$.

Hence, both composite graphs are primitive, although they exhibit different walk and scrambling characteristics. From a theoretical point of view, these findings show that graph operations directly affect primitivity-related parameters. The join produces a highly efficient structure in terms of exponent and scrambling index, whereas for the corona product both parameters remain governed by a structural property of the base graph, namely $\mathrm{diam}(G)$. Therefore, this paper contributes to graph theory by clarifying the relationship between composite graph operations and primitive parameters associated with walk lengths.

Future research may extend these results to other graph products, such as the cartesian product, lexicographic product, and strong product. It would also be interesting to investigate directed graphs, weighted graphs, or other classes of composite graphs in order to obtain a broader understanding of exponent and scrambling index behavior in more general graph structures.

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