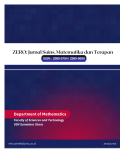
Zero : Jurnal Sains, Matematika, dan Terapan
E-ISSN : 2580-5754; P-ISSN : 2580-569X
Volume 10, Number 1, 2026
DOI: 10.30829/zero.v10i1.27103
Page: 12-24



Exploring the Metric Chromatic Number of Uniform, Centralized Uniform, and Cycle Uniform Theta Graphs

[1] Raventino
Department of Mathematics, Universitas Tanjungpura, Pontianak, Indonesia

[2] Fransiskus Fran
Department of Mathematics, Universitas Tanjungpura, Pontianak, Indonesia

Article Info

Article history:

Accepted 25 January 2026

Keywords:

Chromatic Number;
Distance;
Metric Coloring.

ABSTRACT

Metric coloring allows adjacent vertices of a graph to share the same color provided that their associated distance vectors are distinct, leading to the concept of the metric chromatic number. This notion is closely related to problems of vertex distinguishability and resource allocation in network-like structures. In this paper, we present the first exact determination of the metric chromatic number for three families of theta type graphs: uniform theta graphs, centralized uniform theta graphs, and a newly introduced class called the cycle uniform theta graph, obtained by cyclically arranging uniform theta subgraphs. The proposed construction enables an investigation of how cyclic configurations influence metric coloring behavior. Using a constructive metric coloring approach, exact values of the metric chromatic number are obtained. It is shown that the uniform theta graph $\theta(m,n)$ and the centralized uniform theta graph $\theta(m,n,p)$ both satisfy $\mu\big(\theta(m,n)\big) = \mu(\theta(m,n,p)) = 2$ for all positive integers $m,n,$ and p. For the cycle uniform theta graph $\theta_c(m,n,q)$, the metric chromatic number equals 2 when n and q have the same parity or when n is odd and q is even. In contrast, $\mu\big(\theta_c(m,n,q)\big) = 3$ when n is even and q is odd. This latter case arises because the longest path in the cyclic structure has odd length, forcing the graph to have chromatic number three. Since the graph is connected and its chromatic number is at most three, this structural constraint directly implies that three colors are also necessary for a valid metric coloring.

Corresponding Author:

Raventino,
Department of Mathematics,
Universitas Tanjungpura, Pontianak, Indonesia
Email: raventino@fmipa.untan.ac.id

1. INTRODUCTION

Graph theory provides a versatile framework for modeling relational structures arising in mathematics, computer science, and networked systems. Among the many graph parameters studied, graph coloring occupies a central position due to its deep theoretical foundations and its relevance to problems such as scheduling, resource allocation, and network design [1], [2], [3], [4], [5], [6], [7], [8], [9]. While classical vertex coloring requires adjacent vertices to receive distinct colors, many practical settings demand more flexible labeling schemes that still guarantee vertex distinguishability.

Distance based graph invariants were introduced to address this need for distinguishability beyond adjacency constraints. One of the most influential concepts is the metric dimension, introduced by Harary and Melter [10], which measures the minimum size of a vertex set capable of uniquely identifying all vertices through distance vectors. A comprehensive overview of subsequent developments and recent research directions on the metric dimension can be found in [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. Building on this idea, Chartrand, Okamoto, and Zhang introduced the metric chromatic number [25], which integrates distance based identification with coloring constraints. In a metric coloring, adjacent vertices may share the same color provided that their associated distance vectors are distinct, and the minimum number of colors satisfying this condition is denoted by $\mu(G)$ [26], [27].

Research on the metric chromatic number has encompassed several families of graphs, including pencil graphs, which serve as a basic reference class in the study of metric coloring [10], followed by works on wheel graphs and related constructions [26]. More recent contributions have examined metric coloring on fan type graphs, corona graphs, and graph products, where research on metric coloring is not limited to identifying the graph classes alone, but also aims to reveal how structural features influence coloring behavior. These studies demonstrate that the value of $\mu(G)$ is highly sensitive to structural properties such as symmetry, connectivity, and other inherent graph characteristics [27], [28], [29], [30], [31], [32], [33], [34]. Contemporary surveys further confirm that distance based coloring parameters remain an active area of research, particularly in connection with vertex identification and network labeling problems [35]. Nevertheless, existing results are still largely restricted to relatively simple or well-known graph classes.

Theta type graphs, which are composed of multiple internally disjoint paths connecting a common pair of vertices, have received comparatively limited attention in the context of metric chromatic number. Although these graphs play an important role in both theoretical graph theory and applied network modeling, existing studies have not yet provided a systematic treatment of metric coloring for uniform theta graphs, centralized uniform theta graphs, or cyclic constructions derived from these structures. To the best of our knowledge, explicit determinations of the metric chromatic number for these families are still unavailable in the current literature. The present study is further motivated by recent work of Riyan and Yeni [35] on the total edge irregularity strength of centralized uniform theta graphs, which demonstrates that subtle structural modifications can substantially affect graph labeling parameters. This naturally leads to the question of how such structural variations influence metric coloring behavior, thereby providing a strong rationale for the investigation undertaken in this paper.

Motivated by the framework introduced in [35], this paper focuses on three closely related graph families: the uniform theta graph, the centralized uniform theta graph, and a newly introduced class called the cycle uniform theta graph. The cycle uniform theta graph is constructed by arranging uniform theta subgraphs in a cyclic manner, yielding a hybrid cycle tree structure that enables a systematic investigation of parity effects and cyclic connectivity on metric coloring. The primary objective of this work is to determine the exact metric chromatic numbers of these families and to provide the first complete characterization of $\mu(G)$ for uniform, centralized uniform, and cycle uniform theta graphs. By identifying parameter dependent conditions under which the metric chromatic number changes, this study clarifies the interplay between graph structure, parity, and distance-based coloring constraints, while also offering insights relevant to vertex distinguishability and resource allocation in network-like systems.

2. RESEARCH METHOD

This work is purely theoretical and employs a constructive proof strategy combined with structural graph analysis to determine the metric chromatic numbers of several theta type graph families. The study focuses on uniform theta graphs, centralized uniform theta graphs, and the newly introduced cycle uniform theta graphs. Throughout the paper, all graph parameters are assumed to belong to $\mathbb{N}$, with explicit constraints imposed to ensure that each construction is well defined; in particular, the uniform theta graph $\theta(m,n)$ is considered for $m,n \geq 2$, the centralized uniform theta graph $\theta(m,n,p)$ for $m,n,p \geq 2$, and the cycle uniform theta graph $\theta_c(m,n,q)$ for $m \geq 2$, $n \geq 2$, and $q \geq 3$. This section presents the necessary preliminaries, notation, and proof strategy used consistently in the subsequent analysis.

For each graph family G under consideration, the method proceeds in a systematic manner. First, the classical chromatic number $\chi(G)$ is determined using the structural properties of the graph, which provides a natural upper bound for the metric chromatic number $\mu(G)$. Next, a sequence of candidate colorings is constructed, beginning with two colors and increasing incrementally up to $\chi(G)$. For a fixed number of colors l, an explicit coloring function $f: V(G) \to \{1,2,\dots,l\}$ is defined by specifying the color classes and the associated partition $\pi = \{w_1, w_2, \dots, w_l\}$. For each vertex $v \in V(G)$, the corresponding metric code vector $\Gamma(v,\pi)$ is then computed. A coloring is verified to be a metric coloring by showing that every adjacent pair of vertices has distinct metric code vectors; if this condition is satisfied, the process terminates and the corresponding value of l is identified as $\mu(G)$. Otherwise, the number of colors is increased and the construction is repeated, ensuring that the smallest possible number of colors satisfying the metric coloring condition is obtained.

In the case of cycle uniform theta graphs, the analysis requires a more refined treatment due to the interaction between cyclic and tree-like structures. In particular, the proof is organized into separate cases according to the parity of the parameters n and q, since even–odd configurations significantly affect metric distance patterns and, consequently, the feasibility of two-color metric colorings. Each parity case is addressed explicitly, either through direct distance calculations or via structured lemmas that formalize recurring arguments and avoid reliance on informal analogies. No empirical data are collected, as all results are derived analytically. Finally, for notational convenience, the interval $[a_1, a_2]$ is defined as $[a_1, a_2] = \{ c \in \mathbb{N} \mid a_1 \leq c \leq a_2 \}$ where $a_1, a_2 \in \mathbb{N}$.

In summary, the proof strategy adopted in this paper follows a unified scheme across all graph families under consideration. The chromatic number $\chi(G)$ is first established to obtain a sharp upper bound for $\mu(G)$. Subsequently, explicit metric colorings are constructed, and parity-based case analyses are employed to ensure that all adjacency configurations are covered. The minimality of the resulting colorings is then verified by demonstrating that adjacent vertices admit distinct metric vectors, thereby yielding the exact value of $\mu(G)$ for each graph family.

Definition 1 [10] Consider graph G and $f: V(G) \to \{1,2, \dots, l\}$ be a vertex coloring in which adjacent vertices are allowed to share the same color. Let $\pi = \{w_1, w_2, \dots, w_l\}$ denote the associated collection of color classes, and define for each vertex $v \in V(G)$ the metric vector

$$\Gamma(v, \pi) = (d(v, w_1), d(v, w_2), \dots, d(v, w_l)),$$

where $d(v, w_i) = \min \{d(v, w) \mid w \in w_i\}$. The coloring f is referred to as a metric coloring if every adjacent pair $u, v \in V(G)$ satisfies

$$\Gamma(u, \pi) \neq \Gamma(v, \pi).$$

Definition 2 [26] The quantity arising from identifying the smallest number of colors that can induce a metric coloring on a graph G is referred to as its metric chromatic number, denoted by $\mu(G)$.

Lemma 1 [26] Given a connected graph G with n vertices, the following bound holds for its metric chromatic number:

$$2 \leq \mu(G) \leq \chi(G) \leq n,$$

where $\chi(G)$ is the chromatic number of G.

Definition 3 [35] The uniform theta graph $\theta(m, n)$ for $m, n \in \mathbb{N}$, is formally specified as the graph with vertex set

$$V\big(\theta(m,n)\big) = \{c_0, c_1, v_{ij} \mid i \in [1, m];\ j \in [1, n]\},$$

whose edge set is given by

$$E\big(\theta(m,n)\big) = \Big\{\{c_0, v_{i1}\}, \{v_{in}, c_1\}, \{v_{ij}, v_{i(j+1)}\}\Big|\ i \in [1, m];\ j \in [1, n-1]\}.$$

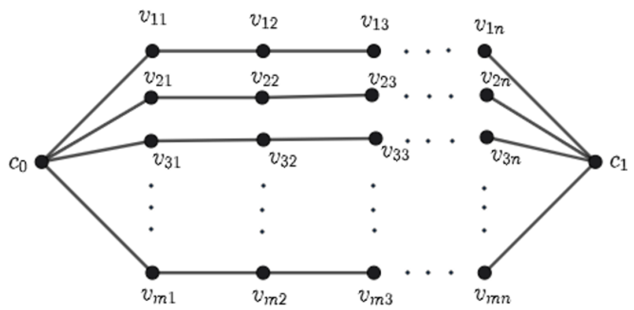


Figure 1. Uniform Theta Graph $\theta(m, n)$

Definition 4 [35] The centralized uniform theta graph $\theta(m, n, p)$ where $m, n, p \in \mathbb{N}$, is defined by the vertex set

$$V\big(\theta(m,n,p)\big) = \{c_0, c_r, v_{ijk} \mid i \in [1, m];\ j \in [1, n];\ r, k \in [1, p]\},$$

whose edge set is given by

$$E\big(\theta(m,n,p)\big) = \Big\{\{c_0, v_{i1k}\}, \{v_{ink}, c_r\}, \{v_{ijk}, v_{i(j+1)k}\}\Big|\ i \in [1, m];\ j \in [1, n-1];\ r, k \in [1, p]\}.$$

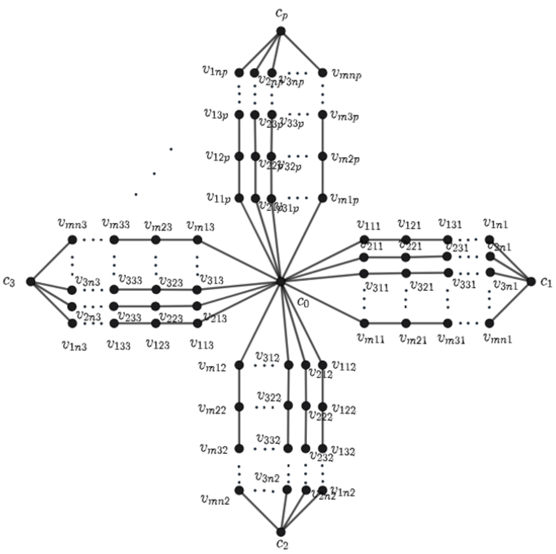


Figure 2. Centralized Uniform Theta Graph $\theta(m,n,p)$

The author then introduces a new graph constructed from the combined ideas of uniform theta graphs and cycle graphs, leading to the following Definition 5.

Definition 5 The cycle uniform theta graph $\theta_c(m,n,q)$ with $m,n,q \in \mathbb{N}$, is characterized by the vertex set

$$V(\theta_c(m,n,q)) = \{c_0, c_r, v_{ijk} \mid i \in [1,m];\ j \in [1,n];\ r \in [1,q-1]; k \in [1,q]\},$$

while its corresponding collection of edges is described by

$$E(\theta_c(m,n,q)) = \left\{\{c_0, v_{i11}\}, \{c_r, v_{i1k}\}, \{v_{inq}, c_0\}, \{v_{ink}, c_r\}, \{v_{ijk}, v_{i(j+1)k}\} \middle|\ i \in [1,m];\ j \in [1,n-1];\ r \in [1,q-1]; k \in [1,q]\right\}.$$

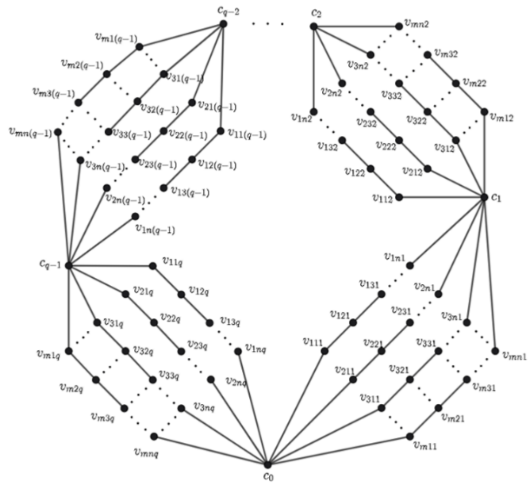


Figure 3. Cycle Uniform Theta Graph $\theta_c(m,n,q)$

Lemma 2 For a uniform theta graph $\theta(m,n)$ with $m,n \in \mathbb{N}$, the chromatic number is

$$\chi(\theta(m,n)) = 2.$$

Proof. Let $\theta(m,n)$ for $m,n \in \mathbb{N}$. When $m = 1$, the graph $\theta(1,n)$ reduces to a path graph. Therefore, $\chi(\theta(1,n)) = 2$. For $m \geq 2$, the construction of $\theta(m,n)$ yields exactly m internally disjoint paths of length n joining the same pair of end vertices. Since these paths have equal length, any two of them form a cycle of even length. Hence, $\theta(m,n)$ contains only even cycles. It follows that $\chi(\theta(m,n)) = 2,$ for all $m \geq 2$. Hence, $\chi(\theta(m,n)) = 2$.

Lemma 3 For the centralized uniform theta graph $\theta(m,n,p)$ with $m,n,p \in \mathbb{N}$, its chromatic number is given by

$$\chi(\theta(m,n,p)) = 2.$$

Proof. Consider $\theta(m,n,p)$ constructed by taking p copies of the graph $\theta(m,n)$ and identifying one prescribed vertex of each copy with a common central vertex c_0. Since each $\theta(m,n)$ is 2-colorable, fix a proper 2-coloring for one copy and apply the same color assignment to all remaining copies. Because the only shared vertex among the $p$copies is c_0, and this vertex receives a consistent color across all copies, no adjacency conflicts arise in the combined structure. Thus, the entire graph admits a proper 2-coloring, and therefore

$$\chi(\theta(m,n,p)) = 2,$$

as claimed.

Lemma 4 Let $\theta_c(m,n,q)$ be a cycle uniform theta graph with $m,n,q \in \mathbb{N}$. Then its chromatic number satisfies

$$\chi\big(\theta_c(m,n,q)\big) = \begin{cases} 2\text{ ; if } n \text{ and } q \text{ are both odd, both even, or if } n \text{ is odd and } q \text{ is even} \\ 3\text{ ; if } n \text{ is even and } q \text{ is odd} \end{cases}$$

Proof. Consider the cycle uniform theta graph $\theta_c(m,n,q)$ with $m,n,q \in \mathbb{N}$. Observe that this graph necessarily contains cycles whose lengths are either even or odd. To determine its chromatic number, it suffices to examine the largest cycle in the graph, since every smaller cycle is structurally analogous to those already characterized in Lemma 2. In the cases where n and q are both odd, both even, or when n is odd and q is even, the largest cycle in $\theta_c(m,n,q)$ has even length. Conversely, when n is even and q is odd, the largest cycle of $\theta_c(m,n,q)$ has odd length. It is well known that an even cycle has chromatic number 2, whereas an odd cycle requires chromatic number 3. Therefore,

$$\chi\big(\theta_c(m,n,q)\big) = \begin{cases} 2\text{ ; if } n \text{ and } q \text{ are both odd, both even, or if } n \text{ is odd and } q \text{ is even} \\ 3\text{ ; if } n \text{ is even and } q \text{ is odd} \end{cases}.$$

3. RESULT AND ANALYSIS

This section presents the main results of the paper concerning the metric chromatic numbers of uniform, centralized uniform, and cycle uniform theta graphs. The results are established through explicit metric coloring constructions and rigorous proofs, and are supported by illustrative examples and figures. In particular, the analysis highlights how structural properties and parity conditions play a decisive role in determining the feasibility and exact value of metric colorings across these graph families.

3.1 Metric Chromatic Number of Uniform Theta Graph

As an illustrative example, consider the graph $\theta(2,3)$, with vertex set

$$V\big(\theta(2,3)\big) = \{c_0, c_1, v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{23}\},$$

and whose edge set is defined as

$$E\big(\theta(2,3)\big) = \{\{c_0, v_{11}\}, \{c_0, v_{21}\}, \{v_{11}, v_{12}\}, \{v_{12}, v_{13}\}, \{v_{21}, v_{22}\}, \{v_{22}, v_{23}\}, \{v_{13}, c_1\}, \{v_{23}, c_1\}\}.$$

To determine the metric chromatic number of the graph $\theta(2,3)$, we first define a function $f: V(\theta(2,3)) \to \{1,2\}$ given by

$$f(v) = \begin{cases} 1, & v = c_0, c_1, v_{12}, v_{22} \\ 2, & v = v_{11}, v_{13}, v_{21}, v_{23} \end{cases}$$

this assignment induces a proper and consistent vertex coloring on the graph $\theta(2,3)$. Based on this coloring, we construct the set of color classes $\pi = \{W_1, W_2\}$, were

$$W_1 = \{c_0, c_1, v_{12}, v_{22}\} \text{ and}$$
$$W_2 = \{ v_{11}, v_{13}, v_{21}, v_{23}\}.$$

Using these color classes, the metric code vector of each vertex is computed with respect to π. The resulting metric code vectors for all vertices of $\theta(2,3)$ are presented in Table 1.

Table 1. Metric code vectors $\Gamma(v,\pi)$ of the graph $\theta(2,3)$

Vertex	$d(v,W_1)$	$d(v,W_2)$	$\Gamma(v,\pi)$
c_0	0	1	(0,1)
c_1	0	1	(0,1)
v_{11}	1	0	(1,0)
v_{12}	0	1	(0,1)
v_{13}	1	0	(1,0)
v_{21}	1	0	(1,0)
v_{22}	0	1	(0,1)
v_{23}	1	0	(1,0)

Based on Table 1, the resulting metric coloring is illustrated in Figure 4.

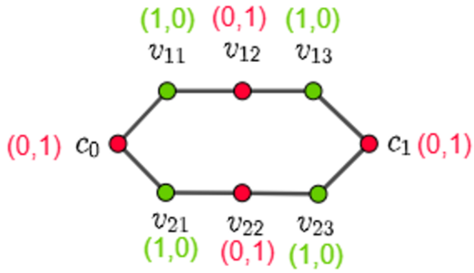


Figure 4. Uniform Theta Graph $\theta(2,3)$

We observe that any two adjacent vertices of $\theta(2,3)$ admit different metric code vectors, which implies that the metric chromatic number of the graph is 2.

In general, for the uniform theta graph $\theta(m,n)$, the metric chromatic number is given in Theorem 1.

Theorem 1 For the uniform theta graph $\theta(m,n)$ where $m,n \in \mathbb{N}$, the metric chromatic number is

$$\mu(\theta(m,n)) = 2.$$

Proof.

Case for odd n

Consider the function $f: V(\theta(m,n)) \to \{1,2\}$ given by

$$f(v) = \begin{cases} 1, v = c_0, c_1, v_{ij} \; with \; i \in [1,m]; \; j \in [1,n]; \; j \; even \\ 2, \qquad v = v_{ij} \; with \; i \in [1,m]; \; j \in [1,n]; \; j \; odd \end{cases}$$

this assignment defines a consistent vertex-coloring on the graph $\theta(m,n)$. Based on this coloring, construct the set of color classes $\pi = \{w_1, w_2\}$, were

$$w_1 = \{ c_0, c_1, v_{ij} | \; i \in [1,m]; \; j \in [1,n]; \; j \; even\}$$

$$w_2 = \{ v_{ij} | \; i \in [1,m]; \; j \in [1,n]; \; j \; odd\}.$$

We analyze the vectors $\Gamma(u,\pi)$ and $\Gamma(v,\pi)$ for all distinct vertices $u, v \in V(\theta(m,n))$ to ensure that each pair of adjacent vertices receives distinct metric representations.

$\Gamma(c_0,\pi) = (d(c_0,w_1), d(c_0,w_2))$. Since

$$d(c_0,w_1) = \min\{d(c_0,c_0), d(c_0,c_1), d(c_0,v_{ij}) | \; i \in [1,m]; \; j \in [1,n]; \; j \; even\} = 0$$

$$d(c_0,w_2) = \min\{ d(c_0,v_{ij}) | \; i \in [1,m]; \; j \in [1,n]; \; j \; odd\} = 1$$

we establish $\Gamma(c_0,\pi) = (0,1)$.

$\Gamma(v_{ij},\pi) = (d(v_{ij},w_1), d(v_{ij},w_2))$ with odd j. Since

$$d(v_{ij},w_1) = \min\{d(v_{ij},c_0), d(v_{ij},c_1), d(v_{ij},v_{it}) | \; i \in [1,m]; \; j,t \in [1,n]; \; t \; even\} = 1$$

$$d(v_{ij},w_2) = \min\{ d(v_{ij},v_{it}) | \; i \in [1,m]; \; j,t \in [1,n]; \; t \; odd\} = 0$$

then $\Gamma(v_{ij},\pi) = (1,0)$ with odd j.

$\Gamma(v_{ij},\pi) = (d(v_{ij},w_1), d(v_{ij},w_2))$ with even j. Since

$$d(v_{ij},w_1) = \min\{d(v_{ij},c_0), d(v_{ij},c_1), d(v_{ij},v_{it}) | \; i \in [1,m]; \; j,t \in [1,n]; \; t \; even\} = 0$$

$$d(v_{ij},w_2) = \min\{ d(v_{ij},v_{it}) | \; i \in [1,m]; \; j,t \in [1,n]; \; t \; odd\} = 1$$

we conclude that $\Gamma(v_{ij},\pi) = (0,1)$ with j even.

$\Gamma(c_1,\pi) = (d(c_1,w_1), d(c_1,w_2))$. Since

$$d(c_1,w_1) = \min\{d(c_1,c_0), d(c_1,c_1), d(c_1,v_{ij}) | \; i \in [1,m]; \; j \in [1,n]; \; j \; even\} = 0$$

$$d(c_1,w_2) = \min\{ d(c_1,v_{ij}) | \; i \in [1,m]; \; j \in [1,n]; \; j \; odd\} = 1$$

then $\Gamma(c_1,\pi) = (0,1)$.

For any adjacent pair of vertices $u, v \in V(\theta(m,n))$ with $u \neq v$, their metric representations satisfy $\Gamma(u,\pi) \neq \Gamma(v,\pi)$. Hence, for odd n, the metric chromatic number of the graph satisfies $\mu(\theta(m,n)) = 2$.

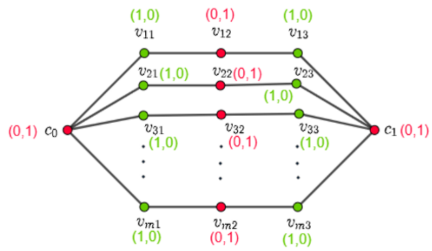


Figure 5. Uniform Theta Graph $\theta(m, 3)$

Case for even n

Consider the function $f: V(\theta(m,n)) \to \{1,2\}$ defined by

$$f(v) = \begin{cases} 1, & v = c_0, v_{ij} \; with \; i \in [1,m]; \; j \in [1,n]; \; j \; even \\ 2, & v = c_1, v_{ij} \; with \; i \in [1,m]; \; j \in [1,n]; \; j \; odd \end{cases}$$

which constitutes a vertex-coloring of the graph $\theta(m,n)$. Define the color classes

$$w_1 = \{ c_0, v_{ij} | \; i \in [1,m]; \; j \in [1,n]; \; j \; even\},$$
$$w_2 = \{ c_1, v_{ij} | \; i \in [1,m]; \; j \in [1,n]; \; j \; odd\},$$

and set of color classes $\pi = \{w_1, w_2\}$.

For each vertex $u \in V(\theta(m,n))$, the corresponding metric vector is

$$\Gamma(u,\pi) = (d(u,w_1), d(u,w_2)),$$

with distances computed as follows:

a. $\Gamma(c_0,\pi) = (0,1)$,
b. $\Gamma(v_{ij},\pi) = (1,0)$ for j odd,
c. $\Gamma(v_{ij},\pi) = (0,1)$ for j even,
d. $\Gamma(c_1,\pi) = (1,0)$.

Since $\Gamma(u,\pi) \neq \Gamma(v,\pi)$ for every pair of adjacent vertices $u \neq v$, f defines a valid metric coloring using two colors, implying $\mu(\theta(m,n)) \leq 2$. By Lemma 2, $\chi(\theta(m,n)) = 2$, and since $\theta(m,n)$ is connected, it follows from [26] result that $\mu(\theta(m,n)) = 2$.

Therefore, for every pair of adjacent vertices in the uniform theta graph, the corresponding metric code vectors are distinct. Hence, we conclude that $\mu\big(\theta(m,n)\big) = 2$, for all $m, n \in \mathbb{N}$.

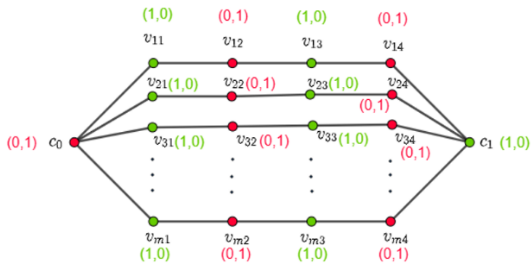


Figure 6. Uniform Theta Graph $\theta(m, 4)$

3.2 Metric Chromatic Number of Centralized Uniform Theta Graph

Theorem 2 Let $\theta(m,n,p)$ denote a centralized uniform theta graph with $m, n, p \in \mathbb{N}$. Then the metric chromatic number of $\theta(m,n,p)$ is

$$\mu(\theta(m,n,p)) = 2.$$

Proof. The proof follows the same constructive framework as that employed for Theorem 1. In the present case, the vertex set is extended from $\{c_0, c_1\}$ to $\{c_0, c_r \mid r \in [1,p]\}$, and each original vertex v_{ij} is replaced by a family of vertices $\{v_{ijk} \mid k \in [1,p]\}$. These extensions do not alter the metric chromatic number of the graph. Consequently, by systematically replacing c_1 with c_r and v_{ij} with v_{ijk} in the argument of Theorem 1, the metric coloring remains valid. Therefore, it follows rigorously that $\mu(\theta(m,n,p)) = 2$.

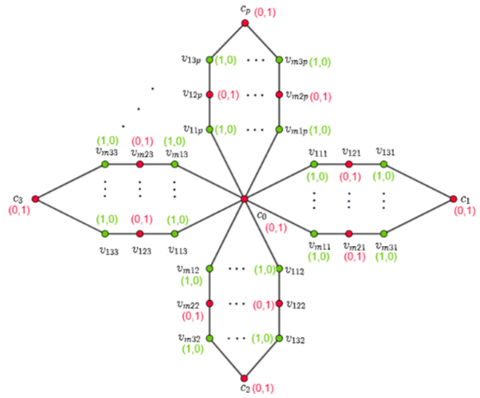


Figure 7. Centralized Uniform Theta Graph $\theta(m, 3, p)$

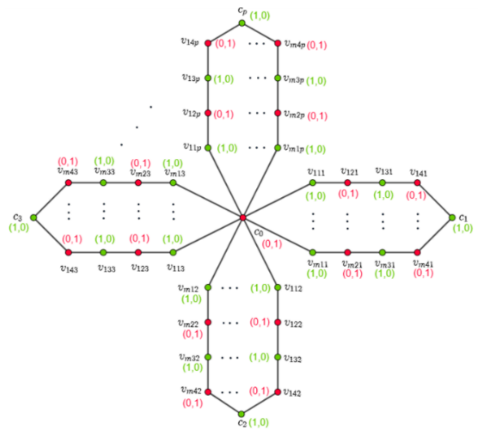


Figure 8. Centralized Uniform Theta Graph $\theta(m, 4, p)$

3.3 Metric Chromatic Number of Cycle Uniform Theta Graphs

Theorem 3 Let cycle uniform theta graph $\theta_c(m, n, q)$ with $m, n, q \in \mathbb{N}$, then

$$\mu(\theta_c(m,n,q)) = \begin{cases} 2, & \text{if } n \text{ and } q \text{ are both odd; both even; or if } n \text{ is odd and } q \text{ is even} \\ 3, & \text{if } n \text{ is even and } q \text{ is odd} \end{cases}$$

Proof.

Case for n and q are both odd

Consider the function $f: V(\theta_c(m,n,q)) \to \{1,2\}$ given by

$$f(v) = \begin{cases} 1, & v = c_0, c_r, v_{ijk} \; with \; i \in [1,m]; \; j \in [1,n]; r \in [1,q-1]; k \in [1,q]; \; j \; even \\ 2, & v = v_{ijk} \; with \; i \in [1,m]; \; j \in [1,n]; k \in [1,q]; \; j \; odd \end{cases}$$

this assignment defines a consistent vertex-coloring on the graph $\theta_c(m,n,q)$. Based on this coloring, construct the set of color classes $\pi = \{W_1, W_2\}$, were

$$W_1 = \{ c_0, c_r, v_{ijk} \mid i \in [1,m]; \; j \in [1,n]; r \in [1,q-1]; k \in [1,q]; \; j \; even\}$$

$$W_2 = \{ v_{ijk} \mid i \in [1,m]; \; j \in [1,n]; k \in [1,q]; \; j \; odd\}$$

we analyze the vectors $\Gamma(u,\pi)$ and $\Gamma(v,\pi)$ for all distinct vertices $u, v \in V(\theta_c(m,n,q))$ to ensure that each pair of adjacent vertices receives distinct metric representations.

$\Gamma(c_0,\pi) = (d(c_0, W_1), d(c_0, W_2))$. Since

$$d(c_0, W_1) = \min\{d(c_0,c_0), d(c_0,c_r), d(c_0, v_{ijk}) \mid i \in [1,m]; \; j \in [1,n]; \; r \in [1,q-1]; \; k \in [1,q]; \; j \; even\} = 0$$

$$d(c_0, W_2) = \min\{ d(c_0, v_{ijk}) \mid i \in [1,m]; \; j \in [1,n]; k \in [1,q]; \; j \; odd\} = 1$$

we conclude that $\Gamma(c_0,\pi) = (0,1)$.

$\Gamma(v_{ijk},\pi) = (d(v_{ijk}, W_1), d(v_{ijk}, W_2))$ with odd j. Since

$$d(v_{ijk}, W_1) = \min\{d(v_{ijk}, c_0), d(v_{ijk}, c_r), d(v_{ijk}, v_{itk}) \mid \; i \in [1,m]; \; j,t \in [1,n]; r \in [1,q-1]; k \in [1,q]; \; t \; even\} = 1$$

$$d(v_{ijk}, W_2) = \min\{ d(v_{ijk}, v_{itk}) \mid \; i \in [1,m]; \; j,t \in [1,n]; k \in [1,q]; \; t \; odd\} = 0$$

then $\Gamma(v_{ijk},\pi) = (1,0)$ with odd j.

$\Gamma(v_{ijk},\pi) = (d(v_{ijk},w_1), d(v_{ijk},w_2))$ with even j. Since

$$d(v_{ijk},w_1) = \min\{d(v_{ijk},c_0), d(v_{ijk},c_r), d(v_{ijk},v_{itk}) \mid i \in [1,m];\ j,t \in [1,n]; r \in [1,q-1]; k \in [1,q];\ t\ even\} = 0$$

$$d(v_{ijk},w_2) = \min\{ d(v_{ijk},v_{itk}) \mid i \in [1,m];\ j,t \in [1,n]; k \in [1,q];\ t\ odd\} = 1$$

we establish $\Gamma(v_{ijk},\pi) = (0,1)$ with even j.

$\Gamma(c_r,\pi) = (d(c_r,w_1), d(c_r,w_2))$. Since

$$d(c_r,w_1) = \min\{d(c_r,c_0), d(c_r,c_r), d(c_r,v_{ijk}) \mid i \in [1,m];\ j \in [1,n]; r \in [1,q-1]; k \in [1,q];\ j\ even\} = 0$$

$$d(c_r,w_2) = \min\{ d(c_r,v_{ijk}) \mid i \in [1,m];\ j \in [1,n]; r \in [1,q-1]; k \in [1,q];\ j\ odd\} = 1$$

then $\Gamma(c_r,\pi) = (0,1)$.

For any adjacent pair of vertices $u,v \in V(\theta_c(m,n,q))$ with $u \neq v$, their metric representations satisfy $\Gamma(u,\pi) \neq \Gamma(v,\pi)$. Hence, for n and q are both odd, the metric chromatic number of the graph satisfies $\mu(\theta_c(m,n,q)) = 2$.

Case for odd n and even q

In this case, the underlying distance structure coincides with that of the odd–odd configuration. The only distinction arises from the increased multiplicity of the vertices c_r, which does not affect the distance relationships to the color classes. Accordingly, assigning each vertex c_r to the color class w_1 preserves distinct metric representations for all adjacent vertices and yields a valid metric coloring.

Case for n and q are both even

This case is divided into two parts, namely when k is odd and when k is even.

For the case where k is odd, the construction is given as follows:

Consider the function $f: V(\theta_c(m,n,q)) \to \{1,2\}$ given by

$$f(v) = \begin{cases} 1, & v = c_0, c_r, v_{ijk}\ with\ i \in [1,m];\ j \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,j\ even \\ 2, & v = c_r, v_{ijk}\ with\ i \in [1,m];\ j \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,j\ odd \end{cases}$$

this assignment defines a consistent vertex-coloring on the graph $\theta_c(m,n,q)$. Based on this coloring, construct the set of color classes $\pi = \{w_1, w_2\}$, were

$$w_1 = \{ c_0, c_r, v_{ijk} \mid i \in [1,m];\ j \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,j\ even\}$$

$$w_2 = \{ c_r, v_{ijk} \mid i \in [1,m];\ j \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,j\ odd\}$$

we analyze the vectors $\Gamma(u,\pi)$ and $\Gamma(v,\pi)$ for all distinct vertices $u,v \in V(\theta_c(m,n,q))$ to ensure that each pair of adjacent vertices receives distinct metric representations.

$\Gamma(c_0,\pi) = (d(c_0,w_1), d(c_0,w_2))$. Since

$$d(c_0,w_1) = \min\{d(c_0,c_0), d(c_0,c_r), d(c_0,v_{ijk}) \mid i \in [1,m];\ j \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,j\ even\} = 0$$

$$d(c_0,w_2) = \min\{ d(c_0,c_r), d(c_0,v_{ijk}) \mid i \in [1,m];\ j \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,j\ odd\} = 1$$

then $\Gamma(c_0,\pi) = (0,1)$.

$\Gamma(v_{ijk},\pi) = (d(v_{ijk},w_1), d(v_{ijk},w_2))$ with odd j. Since

$$d(v_{ijk},w_1) = \min\{d(v_{ijk},c_0), d(v_{ijk},c_r), d(v_{ijk},v_{itk}) \mid i \in [1,m];\ j,t \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,t\ even\} = 1$$

$$d(v_{ijk},w_2) = \min\{ d(v_{ijk},c_r), d(v_{ijk},v_{itk}) \mid i \in [1,m];\ j,t \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,t\ odd\} = 0$$

we conclude that $\Gamma(v_{ijk},\pi) = (1,0)$ with odd j.

$\Gamma(v_{ijk},\pi) = (d(v_{ijk},w_1), d(v_{ijk},w_2))$ with even j. Since

$$d(v_{ijk},w_1) = \min\{d(v_{ijk},c_0), d(v_{ijk},c_r), d(v_{ijk},v_{itk}) \mid i \in [1,m];\ j,t \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,t\ even\} = 0$$

$$d(v_{ijk},w_2) = \min\{d(v_{ijk},c_r), d(v_{ijk},v_{itk}) \mid i \in [1,m];\ j,t \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,t\ odd\} = 1$$

then $\Gamma(v_{ijk},\pi) = (0,1)$ with even j.

$\Gamma(c_r,\pi) = (d(c_r,w_1), d(c_r,w_2))$ with odd r. Since

$$d(c_r,w_1) = \min\{d(c_r,c_0), d(c_r,c_r)\ d(c_r,v_{ijk}) \mid i \in [1,m];\ j \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,j\ even\} = 1$$

$$d(c_r,w_2) = \min\{d(c_r,c_r), d(c_r,v_{ijk}) \mid i \in [1,m];\ j \in [1,n]; r \in [1,q-1]; k \in [1,q];\ r,j\ odd\} = 0$$

we obtain $\Gamma(c_r,\pi) = (1,0)$ with odd r.

$\Gamma(c_r,\pi) = (d(c_r,w_1), d(c_r,w_2))$ with even r. Since

$$d(c_r, w_1) = \min\{d(c_r, c_0), d(c_r, c_r)\ d(c_r, v_{ijk}) \mid i \in [1, m];\ j \in [1, n]; r \in [1, q-1]; k \in [1, q];\ r, j\ even\} = 0$$

$$d(c_r, w_2) = \min\{d(c_r, c_r), d(c_r, v_{ijk}) \mid i \in [1, m];\ j \in [1, n]; r \in [1, q-1]; k \in [1, q];\ r, j\ odd\} = 1$$

then $\Gamma(c_r, \pi) = (0,1)$ with even r.

For the case where k is even, the construction is given as follows:

This case is the same as the scenario when k is odd. The distinction lies in the coloring of each vertex v_{ijk}, where the assignment of colors is interchanged: vertices v_{ijk} with even j that were originally colored with color 1 are now assigned color 1 when j is odd, and conversely, vertices with odd j that were colored with color 2 are now assigned color 2 when j is even.

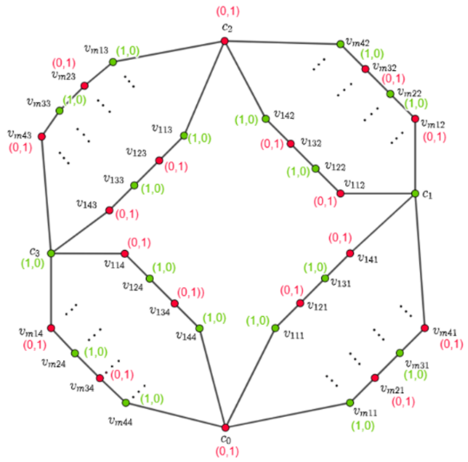


Figure 9. Cycle Uniform Theta Graph $\theta_c(m, 4, 4)$

Case for even n and odd q

In contrast to the even–even case, the parity configuration with n even and q odd prevents the existence of a valid two-color metric coloring. In particular, the vertex v_{inq} admits the same metric code vector as at least one of its adjacent vertices under any two-color assignment. To resolve this conflict, an additional color class w_3 is introduced, with v_{inq} assigned to w_3, which guarantees distinct metric representations for all adjacent vertex pairs. Consequently, the metric coloring of this graph necessitates three distinct colors. For the cases where n and q are odd-odd, odd-even, or even-even, for every pair of adjacent vertices $u, v \in V(\theta_c(m, n, q))$ with $u \neq v$, the corresponding metric vectors satisfy $\Gamma(u, \pi) \neq \Gamma(v, \pi)$. Hence, a metric coloring using two colors is feasible, yielding $\mu(\theta_c(m, n, q)) \leq 2$. By Lemma 4, $\chi(\theta_c(m, n, q)) = 2$, and since $\theta_c(m, n, q)$ is connected, it follows from [26] result that $\mu(\theta_c(m, n, q)) = 2$. Next, in the case where n is even and q is odd, for every adjacent pair $u, v \in V(\theta_c(m, n, q))$ with $u \neq v$, the metric vectors satisfy $\Gamma(u, \pi) \neq \Gamma(v, \pi)$.

Therefore, each pair of adjacent vertices in the cycle uniform theta graph admits distinct metric code vectors. Consequently, a valid metric coloring necessitates three colors, and hence $\mu(\theta_c(m, n, q)) \leq 3$. Applying Lemma 4, we have $\chi(\theta_c(m, n, q)) = 3$, and given that $\theta_c(m, n, q)$ is connected, [26] result implies $\mu(\theta_c(m, n, q)) = 3$.

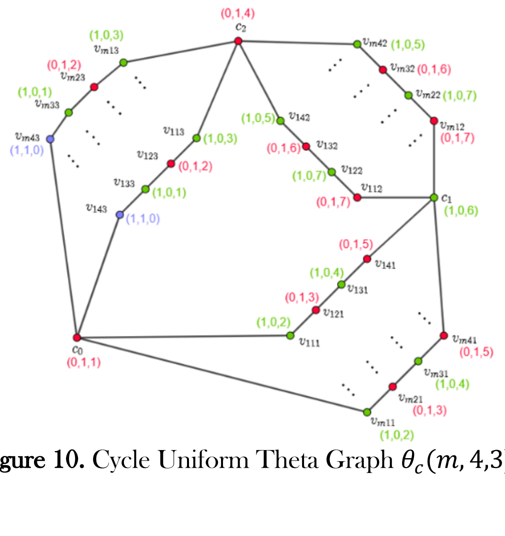


Figure 10. Cycle Uniform Theta Graph $\theta_c(m, 4, 3)$

we establish

$$\mu(\theta_c(m,n,q)) = \begin{cases} 2 & ;\text{if } n \text{ and } q \text{ are both odd, both even, or if } n \text{ is odd and } q \text{ is even} \\ 3 & ;\text{if } n \text{ is even and } q \text{ is odd} \end{cases}$$

for $m, n, q \in \mathbb{N}$.

For clarity and completeness, the main results obtained in this study are summarized in Table 2, which lists the metric chromatic numbers for each theta type graph family under the corresponding parameter conditions.

Table 2. Metric Chromatic Numbers of the Considered Theta Type Graph Families

Graph family	Parameter conditions	Metric chromatic number
uniform theta graph	$m, n \in \mathbb{N}$	$\mu(\theta(m,n)) = 2$
centralized uniform theta graph	$m, n, p \in \mathbb{N}$	$\mu(\theta(m,n,p)) = 2$
cycle uniform theta graph	n and q are both odd, both even, or if n odd and q even, for $m, n, q \in \mathbb{N}$	$\mu(\theta_c(m,n,q)) = 2$
	n even and q odd, for $m, n, q \in \mathbb{N}$	$\mu(\theta_c(m,n,q)) = 3$

These results are consistent with previously studied graph families in metric coloring theory, such as paths, trees, and classical theta graphs, where bipartite or tree-like structures typically yield $\mu(G) = 2$. The cycle uniform theta graph, however, exhibits a clear structural departure from these families, as the interaction between cyclic configurations and parity conditions necessitates the use of a third color in specific cases. From a broader perspective, these findings indicate that parity-sensitive cyclic arrangements play a crucial role in determining metric distinguishability. Beyond their theoretical significance, the results may have implications for the design and analysis of structured networks in which vertex identification based on distance information is essential, such as communication networks, distributed systems, or sensor placement problems. Moreover, the structural insights obtained here suggest potential directions for future work, including the development of efficient metric coloring algorithms for cyclic graph classes and the extension of the analysis to more general network topologies with mixed tree–cycle structures.

4. CONCLUSION

This study provides a complete and rigorous determination of the metric chromatic numbers for several families of theta type graphs. In particular, the results establish that

$$\mu(\theta(m,n)) = 2 \text{ for all } m, n \in \mathbb{N},$$
$$\mu(\theta(m,n,p)) = 2 \text{ for all } m, n, p \in \mathbb{N},$$

and for the cycle uniform theta graph,

$$\mu(\theta_c(m,n,q)) = \begin{cases} 2, & \text{if } n \text{ and } q \text{ are both odd; both even; or if } n \text{ is odd and } q \text{ is even,} \\ 3, & \text{if } n \text{ is even and } q \text{ is odd,} \end{cases}$$

for all $m, n, q \in \mathbb{N}$.

Beyond the exact values, the results reveal a clear structural insight: while uniform and centralized uniform theta graphs admit metric colorings with two colors regardless of parameter parity, the interaction between cyclic structure and path parity in cycle uniform theta graphs introduces a threshold phenomenon that necessitates three colors in the even–odd configuration. This demonstrates that parity and cyclic assemblies play a decisive role in metric distinguishability, even when the underlying graph remains connected and relatively sparse.

The introduction of the cycle uniform theta graph expands the landscape of metric coloring by bridging tree-like theta structures and cyclic constructions, thereby highlighting how combining these two features affects metric coloring behavior. From a broader perspective, these findings contribute to the growing body of work on distance-based graph invariants by illustrating how subtle structural modifications can alter metric chromatic requirements.

Several directions for future research naturally arise from this work. A first problem is to investigate the metric chromatic number of non-uniform theta graphs, where path lengths vary and symmetry is partially broken. Another promising direction is the study of metric chromatic numbers for disconnected graphs or for connected graphs G with chromatic number $\chi(G) \geq 4$, where higher chromatic constraints may interact in nontrivial ways with metric coloring conditions. Such extensions are expected to deepen the understanding of metric coloring on more complex graph families and to uncover new structural phenomena.

5. REFERENCES

[1] B. Omoomi, E. Roshanbin, and M. Vahid Dastjerdi, 'A Polynomial Time Algorithm to Find the Star Chromatic Index of Trees', *The Electronic Journal of Combinatorics*, vol. 28, no. 1, p. P1.6, Jan. 2021, doi: 10.37236/9202.

[2] Z. Dvořák and C. Feghali, 'An Update on Reconfiguring 10-Colorings of Planar Graphs', *The Electronic Journal of Combinatorics*, vol. 27, no. 4, p. P4.51, Dec. 2020, doi: 10.37236/9391.

[3] M. Chudnovsky, A. Kabela, B. Li, and P. Vrána, 'Forbidden Induced Pairs for Perfectness and ω-Colourability of Graphs', *The Electronic Journal of Combinatorics*, vol. 29, no. 2, p. P2.21, May 2022, doi: 10.37236/10708.

[4] S. D. Pasham, 'Network Topology Optimization in Cloud Systems Using Advanced Graph Coloring Algorithms', *Research and Analysis Journal*, vol. 6, no. 11, pp. 01–25, Nov. 2023, doi: 10.18535/raj.v6i11.426.

[5] D. R. Wood, 'Nonrepetitive Graph Colouring', *The Electronic Journal of Combinatorics*, vol. 1000, p. DS24, Sep. 2021, doi: 10.37236/9777.

[6] D. Chawin and I. Haviv, 'On the Subspace Choosability in Graphs', *The Electronic Journal of Combinatorics*, vol. 29, no. 2, p. P2.20, May 2022, doi: 10.37236/10765.

[7] M. Chudnovsky, S. Huang, T. Karthick, and J. Kaufmann, 'Square-Free Graphs with no Induced Fork', *The Electronic Journal of Combinatorics*, vol. 28, no. 2, p. P2.20, May 2021, doi: 10.37236/9144.

[8] C. Antony and S. M. A., 'Star Colouring and Locally Constrained Graph Homomorphisms', *The Electronic Journal of Combinatorics*, vol. 32, no. 3, p. P3.43, Sep. 2025, doi: 10.37236/12949.

[9] N. Kusumastuti, Raventino, and F. Fran, 'The diachromatic number of double star graph', *Journal of Physics: Conference Series*, vol. 2106, no. 1, p. 012024, Nov. 2021, doi: 10.1088/1742-6596/2106/1/012024.

[10] R. Adawiyah, A. Pujiyanto, A. I. Kristiana, D. Dafik, R. M. Prihandini, and S. Susanto, 'Metric Coloring of Pencil Graphs', *JTAM (Jurnal Teori dan Aplikasi Matematika)*, vol. 9, no. 1, p. 68, Jan. 2025, doi: 10.31764/jtam.v9i1.27242.

[11] B. Sooryanarayana and S. Agani Shanmukha, 'Graphs of Neighborhood Metric Dimension Two', *Journal of Mathematical and Fundamental Sciences*, vol. 53, no. 1, pp. 118–133, Jun. 2021, doi: 10.5614/j.math.fund.sci.2021.53.1.9.

[12] S. Prabhu, T. J. Janany, and S. Klavžar, 'Metric dimensions of generalized Sierpiński graphs over squares', *Applied Mathematics and Computation*, vol. 505, p. 129528, Nov. 2025, doi: 10.1016/j.amc.2025.129528.

[13] M. Marsidi, I. H. Agustin, D. Dafik, R. Alfarisi, and H. Siswono, 'On The Metric Dimension of Some Operation Graphs', *CAUCHY: Jurnal Matematika Murni dan Aplikasi*, vol. 5, no. 3, pp. 88–94, Dec. 2018, doi: 10.18860/ca.v5i3.5331.

[14] Z. Shao, S. M. Sheikholeslami, P. Wu, and J.-B. Liu, 'The Metric Dimension of Some Generalized Petersen Graphs', *Discrete Dynamics in Nature and Society*, vol. 2018, pp. 1–10, Aug. 2018, doi: 10.1155/2018/4531958.

[15] S. Nawaz, M. Ali, M. A. Khan, and S. Khan, 'Computing Metric Dimension of Power of Total Graph', *IEEE Access*, vol. 9, pp. 74550–74561, 2021, doi: 10.1109/ACCESS.2021.3072554.

[16] S. Hayat, A. Khan, M. Y. H. Malik, M. Imran, and M. K. Siddiqui, 'Fault-Tolerant Metric Dimension of Interconnection Networks', *IEEE Access*, vol. 8, pp. 145435–145445, 2020, doi: 10.1109/ACCESS.2020.3014883.

[17] V. J. A. Cynthia, M. Ramya, and S. Prabhu, 'Local Metric Dimension of Certain Classes of Circulant Networks', *Journal of Advanced Computational Intelligence and Intelligent Informatics*, vol. 27, no. 4, pp. 554–560, Jul. 2023, doi: 10.20965/jaciii.2023.p0554.

[18] X. Zhang and M. Naeem, 'Metric Dimension of Crystal Cubic Carbon Structure', *Journal of Mathematics*, vol. 2021, pp. 1–8, Jul. 2021, doi: 10.1155/2021/3438611.

[19] S. Prabhu, D. S. R. Jeba, and S. Stephen, 'Metric dimension of star fan graph', *Scientific Reports*, vol. 15, no. 1, p. 102, Jan. 2025, doi: 10.1038/s41598-024-83562-6.

[20] É. Bonnet and N. Purohit, 'Metric Dimension Parameterized By Treewidth', *Algorithmica*, vol. 83, no. 8, pp. 2606–2633, Aug. 2021, doi: 10.1007/s00453-021-00808-9.

[21] J. Muentes, A. J. Becker, A. T. Baraviera, and É. Scopel, 'Metric Mean Dimension and Mean Hausdorff Dimension Varying the Metric', *Qualitative Theory of Dynamical Systems*, vol. 23, no. 1, p. 261, Nov. 2024, doi: 10.1007/s12346-024-01100-1.

[22] S. Nazeer, M. Hussain, A. U. Rehman, and A. H. Ali, 'On metric dimension of symmetrical planer pyramid related graphs', *Afrika Matematika*, vol. 36, no. 1, p. 54, Mar. 2025, doi: 10.1007/s13370-025-01253-5.

[23] L. S. Mhagama, M. F. Nadeem, and M. N. Husin, 'On the edge metric dimension of some classes of cacti', *AIMS Mathematics*, vol. 9, no. 6, pp. 16422–16435, 2024, doi: 10.3934/math.2024795.

[24] S. Nithya and V. Prisci, 'On the metric dimension of extended zero divisor graphs', *Gulf Journal of Mathematics*, vol. 16, no. 1, pp. 205–214, Mar. 2024, doi: 10.56947/gjom.v16i1.1801.

[25] G. Chartrand, F. Okamoto, and P. Zhang, 'The metric chromatic number of a graph', *AUSTRALASIAN JOURNAL OF COMBINATORICS*, vol. 44, pp. 273–286, 2009, doi: 10.5281/zenodo.18027563.
[26] E. Waluyo and A. Fatahillah, 'The Metric Coloring of Related Wheel Graphs', *KADIKMA: Jurnal Matematika dan Pendidikan Matematika*, vol. 13, no. 3, pp. 219–224, 2022, doi: 10.19184/kdma.v13i3.36894.
[27] A. I. Kristiana, A. H. Khusnul, and D. Dafik, 'Locating Metric Coloring on The Cherry Blossom, Sun Flower and Closed Dutch Windmill Graphs', *CAUCHY: Jurnal Matematika Murni dan Aplikasi*, vol. 10, no. 2, pp. 777–789, Aug. 2025, doi: 10.18860/cauchy.v10i2.33636.
[28] M. Y. Rohmatulloh, Slamin, A. I. Kristiana, Dafik, and R. Alfarisi, 'On metric chromatic number of comb product of ladder graph', *Journal of Physics: Conference Series*, vol. 1836, no. 1, p. 012026, Mar. 2021, doi: 10.1088/1742-6596/1836/1/012026.
[29] F. Fujie-Okamoto, W. Renzema, and P. Zhang, 'The k-metric colorings of a graph', *Mathematica Bohemica*, vol. 137, no. 1, pp. 45–63, 2012, doi: 10.21136/MB.2012.142787.
[30] H. Q. Mohammad, S. Haleem. Ibrahem, and L. A. Khaleel, 'The Metric Chromatic Number of Zero Divisor Graph of a Ring Z n', *International Journal of Mathematics and Mathematical Sciences*, vol. 2022, pp. 1–4, Sep. 2022, doi: 10.1155/2022/9069827.
[31] J. Chybowska-Sokół, K. Junosza-Szaniawski, and K. Węsek, 'Coloring distance graphs on the plane', *Discrete Mathematics*, vol. 346, no. 7, p. 113441, Jul. 2023, doi: 10.1016/j.disc.2023.113441.
[32] E. E. Demekhin, A. M. Raigorodskii, and O. I. Rubanov, 'Distance graphs having large chromatic numbers and containing no cliques or cycles of a given size', *Sbornik: Mathematics*, vol. 204, no. 4, pp. 508–538, Apr. 2013, doi: 10.1070/SM2013v204n04ABEH004310.
[33] A. Kupavskiy, 'On the chromatic number of R n with an arbitrary norm', *Discrete Mathematics*, vol. 311, no. 6, pp. 437–440, Mar. 2011, doi: 10.1016/j.disc.2010.12.005.
[34] J. I. Brown, 'The complexity of generalized graph colorings', *Discrete Applied Mathematics*, vol. 69, no. 3, pp. 257–270, Aug. 1996, doi: 10.1016/0166-218X(96)00096-0.
[35] R. W. Putra and Y. Susanti, 'On total edge irregularity strength of centralized uniform theta graphs', *AKCE International Journal of Graphs and Combinatorics*, vol. 15, no. 1, pp. 7–13, Apr. 2018, doi: 10.1016/j.akcej.2018.02.002.