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Closed-Form Formulas for Fibonacci Numbers of Broom and Double Star Graphs

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ABSTRACT

Let $G = (V(G), E(G))$ be a graph, and let $i(G)$ denote the number of independent sets of G, commonly known as the Fibonacci number of the graph. This paper investigates the Fibonacci numbers of two graph families, namely broom graphs and double star graphs, which naturally generalize the classical path and star graphs. By employing a combinatorial approach based on the systematic enumeration of independent sets, we establish recurrence relations governing these invariants. In particular, the Fibonacci number of the broom graph $B_{n,m}$ satisfies $i(B_{n,m}) = i(B_{n-1,m}) + i(B_{n-2,m})$ under appropriate initial conditions, while the Fibonacci number of the double star graph $S_{n,m}$ satisfies $i(S_{n,m}) = i(S_{n-1,m}) + 2^{n-1} \times i(S_{0,m})$. Solving these recurrences yields explicit closed-form expressions for both graph families. To the best of our knowledge, such unified recurrence-based characterizations and closed formulas for the Fibonacci numbers of broom and double star graphs have not been previously reported. These results clarify how the structural features of these graphs influence the growth behavior of their Fibonacci numbers and enrich the study of Fibonacci type invariants in graph theory.

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1. INTRODUCTION

The Fibonacci sequence F_n is a classical recursive sequence defined by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with initial values $F_0 = 0$ and $F_1 = 1$ [1], [2]. Beyond its simple algebraic definition, the Fibonacci sequence arises naturally in a wide range of growth phenomena and recursive processes, and it has long served as a fundamental model in mathematics and combinatorics [3], [4]. A graph G is defined as an ordered pair $(V(G), E(G))$, where $V(G)$ is a nonempty set of vertices and $E(G)$ is a (possibly empty) set of edges, each consisting of an unordered pair of distinct vertices in $V(G)$ [5], [6], [7], [8], [9], [10], [11]. The connection between Fibonacci numbers and graph theory was formally introduced in [12] through the notion of the Fibonacci number of a graph G, denoted by $i(G) = |I(G)|$, where $I(G)$ represents the collection of all independent vertex sets of G. Since then, this invariant has been widely studied as a means of capturing combinatorial and structural properties of graphs, with applications in enumeration problems, network modeling, and algorithmic analysis.

A substantial body of research has focused on determining Fibonacci numbers for various graph classes, including run graphs, cubic graphs, and tricyclic graphs [13], [14], [15], [16], [17], [18], [19], [20], [21]. In addition, explicit formulas have been obtained for several fundamental graph families. In particular, it was shown in [22] that F_{n+2} coincides with the Fibonacci number of the path graph P_n, while [23] established that the Fibonacci number of the star graph S_m equals 2^{m+1}. These classical results demonstrate how simple graph topologies give rise to transparent independent set structures and closed-form expressions.

More recently, attention has shifted toward composite and hybrid graph families formed by combining basic components, such as paths, cycles, and stars. Examples include tadpole graphs, lollipop graphs, and other unicyclic or bicyclic extensions, for which Fibonacci type invariants have been partially characterized [24], [25], [26], [27]. Within this context, broom graphs and double star graphs arise as natural yet structurally distinct generalizations. The broom graph $B_{n,m}$ is obtained by attaching the pendant vertex of a path graph P_n to the central vertex of a star graph S_m [28], while the double star graph $S_{n,m}$ is formed by connecting the centers of two star graphs S_n and S_m [29], [30]. Unlike other composite graphs, these families simultaneously exhibit elongated path components and pronounced branching structures, leading to more intricate interactions among independent sets.

Although the structural and spectral properties of broom and double star graphs have been investigated in previous studies, their Fibonacci numbers have not yet been fully characterized. In contrast to path and star graphs, whose independent set configurations admit well-known closed formulas, broom and double star graphs display more complex combinatorial behavior due to the interplay between linear and branching substructures. As a result, explicit recurrence relations and closed-form expressions describing the growth of independent sets in these graphs remain incomplete in the existing literature.

Motivated by the lack of a comprehensive characterization, the present study provides a systematic characterization of the Fibonacci numbers associated with the broom graph $B_{n,m}$ and the double star graph $S_{n,m}$. By analyzing how independent sets propagate across path and star components, we derive explicit recurrence relations governing $i(B_{n,m})$ and $i(S_{n,m})$, and subsequently obtain corresponding closed-form formulas. In particular, it is shown that the Fibonacci number of $B_{n,m}$ satisfies a second order Fibonacci type recurrence, while the Fibonacci number of $S_{n,m}$ obeys a recurrence involving both linear and exponential terms. These results extend classical findings for path and star graphs to more complex hybrid structures and contribute to a deeper understanding of Fibonacci type graph invariants, which are relevant to combinatorial theory, graph algorithms, and the modeling of hierarchical and branching networks.

The remainder of this paper is organized as follows. Section 2 describes the theoretical framework and combinatorial methodology used to enumerate independent sets and derive recurrence relations. Section 3 presents the main results, including the derivation and verification of recurrence relations and closed-form expressions for the Fibonacci numbers of $B_{n,m}$ and $S_{n,m}$. Section 4 concludes the paper and outlines directions for future research. References are listed in Section 5.

2. RESEARCH METHOD

This study is purely theoretical and is conducted within a combinatorial framework in graph theory to determine the Fibonacci numbers of the broom graph $B_{n,m}$ and the double star graph $S_{n,m}$. No empirical data collection is involved. The analysis is based on the systematic enumeration of independent sets and the identification of recursive structures induced by graph topology.

Let $G = (V(G), E(G))$ be a finite simple graph with $|V(G)| = k$. Each independent set of G can be represented by a binary string $x = x_1x_2 \cdots x_k$, $x_i \in \{0,1\}$ where the i-th entry corresponds to the vertex $v_i \in V(G)$. The entry $x_i = 1$ indicates that the vertex v_i is included in the independent set, while $x_i = 0$ indicates that v_i is excluded. A binary string is said to be admissible if no two entries equal to 1 correspond to adjacent vertices in G. The collection of all admissible binary strings representing independent sets of G is denoted by $N_b(G)$. Consequently, the Fibonacci number of the graph G is given by $i(G) = |N_b(G)| = |I(G)|$. This binary representation provides a structured and transparent framework for partitioning independent sets and identifying recursive patterns across related graph families.

The binary encoding framework is first applied to the path graph P_n and the star graph S_m, which serve as fundamental components for the graph families under consideration. For the broom graph $B_{n,m}$, independent sets are analyzed by explicitly partitioning cases according to the terminal vertex of the path component. Assume first that this terminal vertex is excluded. In this case, the admissible binary strings are in bijective correspondence with those in $N_b(B_{n-1,m})$. Alternatively, assume that the terminal vertex is included. Then its adjacent vertex must be excluded, and the remaining admissible configurations correspond bijectively to those in $N_b(B_{n-2,m})$. This explicit case distinction leads directly to a second order recurrence relation governing $i(B_{n,m})$, reflecting the recursive structure inherited from the path component together with the branching contribution of the star component.

For the double star graph $S_{n,m}$, independent sets are similarly examined by partitioning admissible binary strings according to the inclusion or exclusion of the central vertices. Assume first that a central vertex is excluded; the resulting admissible configurations correspond to those of a reduced double star graph. Assume next that a central vertex is included. In this case, all adjacent leaves must be excluded, and the remaining choices yield an exponential contribution determined by the independent sets of the opposite star component. By explicitly stating these assumptions at the outset of each case, a recurrence relation combining linear and exponential terms is obtained, capturing the interaction between the two branching structures.

The derived recurrence relations are solved using standard techniques for linear recurrences, with initial conditions obtained from small graph instances. Explicit computations for selected values of n and m are carried out to verify that the proposed recurrences and closed-form formulas agree with direct enumeration, thereby confirming the internal consistency and correctness of the theoretical framework.

3. RESULT AND ANALYSIS

An independent set of a graph $G = (V(G), E(G))$ is a subset $S(G) \subseteq V(G)$ such that no two vertices in $S(G)$ are adjacent. The collection of all independent sets of a graph G is denoted by $I(G)$. The empty set $\emptyset$ is also included in this collection, which can be expressed as $\emptyset \in I(G)$ [31]. The cardinality of the collection of independent sets of a graph G is represented by $|I(G)|$. Consequently, the Fibonacci number of a graph G is defined as $i(G) = |I(G)|$, where $I(G)$ denotes the collection of all independent sets of G [24].

3.1 Fibonacci Numbers of Path Graph P_n and Star Graph S_m

The collection of all independent sets in the path graph $I(P_n)$ and the collection of all independent sets in the star graph $I(S_m)$ can be represented in a binary numbers 0 and 1 [31], binary number 0 represents adjacent nodes while a binary number 1 represents non-adjacent nodes, then the number of binary numbers in a set will be equal to the graph nodes. Furthermore, the set $I(P_n)$ in binary form can be written as $N_b(P_n)$ and the set $I(S_m)$ in binary form can be written as $N_b(S_m)$.

Example 1 Given a path graph P_n represented as in Figure 1

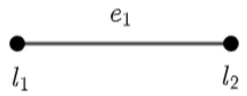

Figure 1. Path graph P_2 with $|V(P_2)| = 2$ and $|E(P_2)| = 1$

Based on Figure 1, the collection of all independent sets of path graph P_2 is $I(P_2) = \{\emptyset, \{l_1\}, \{l_2\}\}$. By encoding each independent set in binary form, we obtain $\emptyset = 00$, $\{l_1\} = 10$, and $\{l_2\} = 01$. Hence, the binary representation of $I(P_2)$ is $N_b(P_2) = \{00,10,01\}$.

Lemma 1 The collection number of all independent sets on a path graph in binary form is $|N_b(P_n)| = |N_b(P_{n-1})| + |N_b(P_{n-2})|$.

Proof. Consider the set $N_b(P_n)$, which can be partitioned into two subsets: $N_0(P_n)$, consisting of all binary strings whose last digit is 0, and $N_1(P_n)$, consisting of all binary strings whose last digit is 1. Thus,

$$N_b(P_n) = N_0(P_n) \cup N_1(P_n).$$

For each $x \in N_0(P_n)$, the ending digit of x is 0. Removing this digit produces a shorter string $x' \in N_b(P_{n-1})$. Conversely, attaching a final digit 0 to every element of $N_b(P_{n-1})$ generates every element of $N_0(P_n)$. Hence,

$$|N_0(P_n)| = |N_b(P_{n-1})|.$$

Similarly, every $x \in N_1(P_n)$ ends with the digit 1. Eliminating the last position produces a string $x' \in N_b(P_{n-2})$, and attaching 1 at the end of each element of $N_b(P_{n-2})$ generates exactly the strings in $N_1(P_n)$. Therefore,

$$|N_1(P_n)| = |N_b(P_{n-2})|.$$

Since

$$N_b(P_n) = N_0(P_n) \cup N_1(P_n),$$

we obtain the recurrence

$$|N_b(P_n)| = |N_b(P_{n-1})| + |N_b(P_{n-2})|.$$

Theorem 1 [22] Let P_n be a path graph, then $i(P_n) = F_{n+2}$.
Proof. Based on Lemma 1, the Fibonacci number of the path graph P_n is

$$\begin{aligned} i(P_n) &= |I(P_n)| \\ &= |N_b(P_n)| \\ &= i(N_b(P_{n-1})) + i(N_b(P_{n-2})) \\ &= F_{(n-1)+2} + F_{(n-2)+2} \\ &= F_{n+1} + F_n \\ &= F_{n+2}. \end{aligned}$$

Hence, it is proved that the Fibonacci number of the path graph P_n is $i(P_n) = F_{n+2}$.
Based on Example 1 and Theorem 1, as follows

$$\begin{aligned} i(P_2) &= |I(P_2)| \\ &= F_{2+2} \\ &= F_4 \\ &= F_3 + F_2 \\ &= 2 + 1 \\ &= 3. \end{aligned}$$

Example 2 Given a star graph S_2 represented as Figure 2.

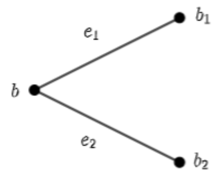

Figure 2. Star graph S_2 with $|V(S_2)| = 3$ and $|E(S_2)| = 2$

Based on Figure 2 the collection of all independent sets in the star graph S_2 is $I(S_2) = \{\emptyset, \{b\}, \{b_1\}, \{b_2\}, \{b_1, b_2\}\}$, then the set $I(S_2)$ in binary form is $N_b(S_2) = \{000,100,010,001,011\}$.
The collection of all independent sets on a star graph in binary form $N_b(S_m)$ can be partitioned into several sets, we given

$$\begin{aligned} N_0(S_m) &= \{000 \dots 0\} \\ N_1(S_m) &= \{100 \dots 0, \dots, 000 \dots 01\} \\ N_2(S_m) &= \{0110 \dots 0, \dots, 000 \dots 011\} \\ N_3(S_m) &= \{01110 \dots 0, \dots, 000 \dots 0111\} \\ &\vdots \\ N_m(S_m) &= \{0111 \dots 1\} \end{aligned}$$

such that the set $N_0(S_m) \cup N_1(S_m) \cup N_2(S_m) \cup N_3(S_m) \cup, \dots, \cup N_m(S_m) = N_b(S_m)$.
The set $N_0(S_m)$ consists of elements that are represented solely by the binary digit 0. On the other hand, the sets

$$N_1(S_m), N_2(S_m), N_3(S_m), \dots, N_m(S_m)$$

contain elements represented using binary digits 0 and 1. By enumerating the elements belonging to each of the sets $N_1(S_m), N_2(S_m), \dots, N_m(S_m)$, their cardinalities can be determined explicitly and expressed in terms of binomial coefficients as follows

$$\begin{aligned} |N_0(S_m)| &= C(m, 0) \\ |N_1(S_m)| &= C(m, 1) + 1 \\ |N_2(S_m)| &= C(m, 2) \\ |N_3(S_m)| &= C(m, 3) \\ &\vdots \\ |N_m(S_m)| &= C(m, m). \end{aligned}$$

Furthermore, since the set $\bigcup_{i=0}^{m} N_i(S_m) = N_b(S_m)$, then by adding the number of each set $N_i(S_m)$, $i = 1,2,\dots,m$ is obtained

$$
\begin{aligned}
|N_b(S_m)| &= \sum_{i=0}^{m} |N_i(S_m)| \\
&= C(m,0) + C(m,1) + 1 + C(m,2) + C(m,3) + \cdots + C(m,m) \\
&= (C(m,0) + C(m,1) + C(m,2) + C(m,3) + \cdots + C(m,m)) + 1 \\
&= \left(\sum_{i=0}^{m} C(m,i)\right) + 1 \\
&= 2^m + 1.
\end{aligned}
$$

Hence, the number of sets $N_b(S_m)$ is $|N_b(S_m)| = 2^m + 1$.

Theorem 2 [23] Given S_m be a star graph, then $i(S_m) = 2^m + 1$.

Proof. The Fibonacci number of a graph is the number of collections of all independent sets in the graph, then the Fibonacci number of the star graph $i(S_m)$ is

$$
\begin{aligned}
i(S_m) &= |I(S_m)| \\
&= |N_b(S_m)| \\
&= 2^m + 1.
\end{aligned}
$$

Hence, it is proved that the Fibonacci number of the star graph S_m is $i(S_m) = 2^m + 1$.

Based on Example 2 and Theorem 2, $i(S_2)$ can be obtained, as follows

$$
\begin{aligned}
i(S_2) &= |I(S_2)| \\
&= 2^2 + 1 \\
&= 5.
\end{aligned}
$$

3.2 Fibonacci Numbers on Broom Graph $B_{n,m}$

The Fibonacci number associated with a broom graph, denoted $i(B_{n,m})$, is determined by analyzing the structural characteristics of $B_{n,m}$ in order to derive a closed form formula for $|I(B_{n,m})|$. Once this expression is established, the value of $i(B_{n,m})$ follows as a direct consequence. To initiate the derivation of $|I(B_{n,m})|$, we consider the fundamental case of the broom graph $B_{0,m}$, where $n = 0$ and $m \geq 2$. Further see subsection 3.2.1.

3.2.1 Broom Graph $B_{0,m}$

The broom graph $B_{0,m}$ corresponds to the case where $n = 0$ and $m \geq 2$. As an illustration of this base structure, the graph $B_{0,2}$ is presented in Figure 3.

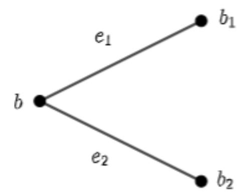

Figure 3. Broom graph $B_{0,2}$

Based on Figure 3 the broom graph $B_{0,2} \cong S_2$ consequently $|I(B_{0,2})| = |I(S_2)|$ so the Fibonacci number of the broom graph $B_{0,2}$ is $i(B_{0,2}) = 2^2 + 1 = 5$.

Consequently, applying this procedure for computing $|I(B_{0,m})|$ yields the Fibonacci number of the broom graph $B_{0,m}$, expressed as $i(B_{0,m}) = 2^m + 1$.

3.2.2 Broom Graph $B_{1,m}$

For the case $n = 1$ and $m \geq 2$, the resulting broom graph is denoted by $B_{1,m}$. An example of this structure, $B_{1,2}$ is depicted in Figure 4.

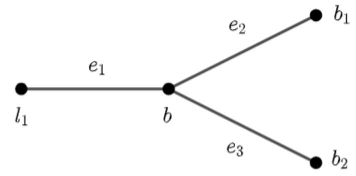

Figure 4. Broom graph $B_{1,2}$

Based on Figure 4 the broom graph $B_{1,2} \cong S_{2+1}$ consequently $|I(B_{1,2})| = |I(S_{2+1})|$, then the Fibonacci number of the broom graph $B_{1,2}$ is $i(B_{1,2}) = 2^{2+1} + 1 = 9$.
By applying the same method to find each $|I(B_{1,m})|$, the Fibonacci number of broom graph $B_{1,m}$ is $i(B_{1,m}) = 2^{m+1} + 1$.

3.2.3 Broom Graph $B_{2,m}$

For $n = 2$ and $m \geq 2$, the corresponding structure is referred to as the broom graph $B_{2,m}$. For illustrative purposes, the specific instance $B_{2,2}$ is presented in Figure 5.

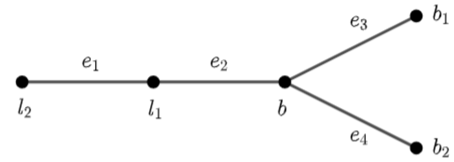

Figure 5. Broom graph $B_{2,2}$

Based on Figure 5, the collection of all independent sets in the broom graph $B_{2,2}$ is $I(B_{2,2}) = I(B_{1,2}) \cup \{\{l_2\}, \{l_2, b\}, \{l_2, b_1\}, \{l_2, b_2,\}, \{l_2, b_1, b_2\}\}$, then that many sets of $I(B_{2,2})$ can be obtained

$$\begin{aligned} |I(B_{2,2})| &= |I(B_{1,2})| + C(2,0) + 1 + C(2,1) + C(2,2) \\ &= 9 + 1 + 1 + 2 + 1 \\ &= 14. \end{aligned}$$

By applying the same method to find each $|I(B_{2,m})|$, the broom graph pattern of $B_{2,m}$ is obtained which forms a formula, namely

$$\begin{aligned} |I(B_{2,m})| &= |I(B_{1,m})| + C(m,0) + 1 + C(m,1) + C(m,2) + \cdots + C(m,m) \\ &= 2^{m+1} + 1 + \big(C(m,0) + C(m,1) + C(m,2) + \cdots + C(m,m)\big) + 1 \\ &= 2^{m+1} + 1 + \left(\sum_{i=0}^{m} C(m,i)\right) + 1 \\ &= 2^{m+1} + 1 + 2^m + 1. \end{aligned}$$

From the formula $|B_{2,m}|$, we can obtain the Fibonacci number of the graph $B_{2,m}$,

$$\begin{aligned} i(B_{2,m}) &= |I(B_{1,m})| \\ &= 2^{m+1} + 1 + 2^m + 1 \\ &= i(B_{1,m}) + i(B_{0,m}). \end{aligned}$$

Hence, the Fibonacci number of broom graph $B_{2,m}$ is $i(B_{2,m}) = i(B_{1,m}) + i(B_{0,m})$.

3.2.4 Broom Graph $B_{3,m}$

Let $n = 3$ and $m \geq 2$. The resulting broom graph is denoted by $B_{3,m}$, with $B_{3,2}$ depicted in Figure 6 as a representative example.

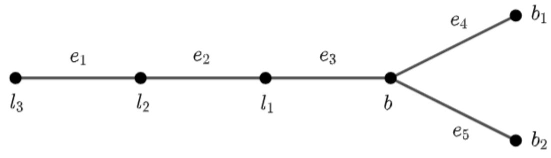

Figure 6. Broom graph $B_{3,2}$

Based on Figure 6, the collection of all independent sets in the broom graph $B_{3,2}$, is $I(B_{3,2}) = I(B_{2,2}) \cup \{\{l_3\},\{l_3,l_1\},\{l_3,b\},\{l_3,b_1\},\{l_3,b_2\},\{l_3,l_1,b_1\},\{l_3,l_1,\ b_2\},\{l_3,b_1,b_2\},\{l_3,l_1,b_1,b_2\}\}$ thus, the collection of all independent sets of $I(B_{3,2})$ is obtained as follows

$$\begin{aligned} |I(B_{3,2})| &= |I(B_{2,2})| + C(3,0) + C(3,1) + 1 + C(3,2) + C(3,3) \\ &= 14 + 1 + 3 + 1 + 3 + 1 \\ &= 23. \end{aligned}$$

Through the analogous computation of $|I(B_{3,m})|$, the combinatorial structure of $B_{3,m}$ can be captured by the formula

$$\begin{aligned} |I(B_{3,m})| &= |I(B_{2,m})| + C(m+1,0) + C(m+1,1) + 1 + C(m+1,2) + \cdots + C(m+1,m+1) \\ &= |I(B_{2,m})| + (C(m+1,0) + C(m+1,1) + C(m+1,2) + \cdots + C(m+1,m+1)) + 1 \\ &= |I(B_{2,m})| + \left(\sum_{i=0}^{m+1} C(m+1,i)\right) + 1 \\ &= 2^m(3) + 2 + 2^{m+1} + 1 \\ &= 2^m(3) + 2 + 2^{m+1} + 1. \end{aligned}$$

Furthermore, from the formula $|I(B_{3,m})|$, we can obtain the Fibonacci number on the graph $B_{3,m}$ as follows

$$\begin{aligned} i(B_{3,m}) &= |I(B_{3,m})| \\ &= 2^m(3) + 2 + 2^{m+1} + 1 \\ &= i(B_{2,m}) + i(B_{1,m}). \end{aligned}$$

Hence, the Fibonacci number of broom graph $B_{3,m}$ is $i(B_{3,m}) = i(B_{2,m}) + i(B_{1,m})$.
Using the same procedure to compute $|I(B_{n,m})|$, the Fibonacci numbers of the broom graphs $B_{n,m}$ are summarized in Table 1.

Table 1. Formula of $|I(B_{n,m})|$, with $n \geq 0$ and $m \geq 2$

No	Type of Graph	$i(B_{n,m})$
1.	$B_{0,m}$	$i(S_m)$
2.	$B_{1,m}$	$i(S_{m+1})$
3.	$B_{2,m}$	$i(B_{1,m}) + i(B_{0,m})$
⋮	⋮	⋮
n.	$B_{n,m}$	$i(B_{n-1,m}) + i(B_{n-2,m})$

Theorem 3 Let $B_{n,m}$ be a broom graph, then $i(B_{n,m}) = i(B_{n-1,m}) + i(B_{n-2,m})$.
Proof. Let $N_b(B_{n,m})$ denote the collection of all independent sets of a broom graph represented in binary form. This set can be partitioned into two subsets: $N_0(B_{n,m})$, whose elements begin with 0, and $N_1(B_{n,m})$, whose elements begin with 1, such that $N_0(B_{n,m}) \cup N_1(B_{n,m}) = N_b(B_{n,m})$. For any $x \in N_0(B_{n,m})$, its binary representation begin with 0. Removing this first bit produces a string $x' \in N_b(B_{n-1,m})$. Conversely, appending 0 to every element of $N_b(B_{n-1,m})$ yields exactly the elements of $N_0(B_{n,m})$. Thus, $|N_0(B_{n,m})| = i(B_{n-1,m})$. Similarly, for any $x \in N_1(B_{n,m})$, its binary representation begin with 10. By deleting this first two bit, we obtain $x' \in N_b(B_{n-2,m})$. Conversely, appending 10 to each element of $N_b(B_{n-2,m})$ generates all elements of $N_1(B_{n,m})$. Hence, $|N_1(B_{n,m})| = i(B_{n-2,m})$. Since

$$N_0(B_{n,m}) \cup N_1(B_{n,m}) = N_b(B_{n,m}),$$

it follows that

$$i(B_{n,m}) = i(B_{n-1,m}) + i(B_{n-2,m}).$$

Therefore, the recurrence relation

$$i(B_{n,m}) = i(B_{n-1,m}) + i(B_{n-2,m})$$

is established.
Corollary 1 Given $B_{n,m}$ be a broom graph, then

$$i(B_{n,m}) = \frac{(2^m\times\sqrt{5}+3\times2^m+\sqrt{5}+1)(1+\sqrt{5})^n+(2^m\times\sqrt{5}-3\times2^m+\sqrt{5}-1)(1-\sqrt{5})^n}{2^{n+1}\times\sqrt{5}}.$$

Proof. Given the recursive relation for the Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2},$$

assume that $F_n = i(B_{n,m})$. Then the characteristic equation associated with

$$F_n = F_{n-1} + F_{n-2}$$

is

$$r^2 - r - 1 = 0.$$

Thus, the characteristic equation has two distinct roots:

$$r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}.$$

Since the roots differ, the general homogeneous solution can be expressed as

$$F_n = C_1(\frac{1+\sqrt{5}}{2})^n + C_2(\frac{1-\sqrt{5}}{2})^n.$$

Next, using the initial conditions

$$F_0 = i(B_{0,m}) = 2^m + 1,$$
$$F_1 = i(B_{1,m}) = 2^{m+1} + 1 = 2 \cdot 2^m + 1,$$

we obtain the following system:

$$i(B_{0,m}) = C_1(\frac{1+\sqrt{5}}{2})^0 + C_2(\frac{1-\sqrt{5}}{2})^0 \Rightarrow 2^m + 1 = C_1 + C_2, \quad (1)$$

$$i(B_{1,m}) = C_1(\frac{1+\sqrt{5}}{2}) + C_2(\frac{1-\sqrt{5}}{2}) \Rightarrow 2 \cdot 2^m + 1 = \frac{C_1 + C_2}{2} + \frac{\sqrt{5}}{2}(C_1 - C_2). \quad (2)$$

Solving equations (1) and (2) yields:

$$C_1 = 2^{m-1} + \frac{3 \cdot 2^{m-2}}{\sqrt{5}} + \frac{1}{2} + \frac{1}{2\sqrt{5}}, C_2 = 2^{m-1} - \frac{3 \cdot 2^{m-2}}{\sqrt{5}} + \frac{1}{2} - \frac{1}{2\sqrt{5}}.$$

Since $F_n = i(B_{n,m})$, we deduce the general solution:

$$i(B_{n,m}) = C_1(\frac{1+\sqrt{5}}{2})^n + C_2(\frac{1-\sqrt{5}}{2})^n.$$

Hence,

$$i(B_{n,m}) = \frac{(2^m\sqrt{5} + 3 \cdot 2^m + \sqrt{5} + 1)(1+\sqrt{5})^n + (2^m\sqrt{5} - 3 \cdot 2^m + \sqrt{5} - 1)(1-\sqrt{5})^n}{2^{n+1}\sqrt{5}}. \blacksquare$$

3.3 Fibonacci Numbers on Double Star Graph $S_{n,m}$

The Fibonacci number of the double star graph $i(S_{n,m})$ can be obtained by analyzing the structural pattern of the double star graph $S_{n,m}$, followed by establishing the formula for $|I(S_{n,m})|$. From this formula, the value of $i(S_{n,m})$ is then determined. As an initial step in deriving the formula for $|I(S_{n,m})|$, we begin by considering the double star graph $S_{0,m}$. Further explanation regarding this case is presented in Section 3.3.1.

3.3.1 Double Star Graph $S_{0,m}$

The double star graph $S_{0,2}$ is obtained by combining the star graph S_n with $n = 0$ and the star graph S_m with $m = 2$. In particular, the double star $S_{0,2}$ is shown in Figure 7.

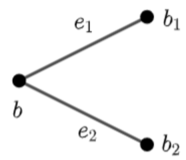

Figure 7. Double star graph $S_{0,2}$

Based on Figure 7 the double star graph $S_{0,2} \cong S_2$ consequently $|I(S_{0,2})| = |I(S_2)|$ so the Fibonacci number of the double star graph $S_{0,2}$ is $i(S_{0,2}) = 2^2 + 1$.

By applying the same method to find each $I(S_{0,m})$ the Fibonacci number of the double star graph $S_{0,m}$ is obtained, $i(S_{0,m}) = 2^m + 1$.

3.3.2 Double Star Graph $S_{1,m}$

The double star graph $S_{1,2}$ is the graph constructed by the star graph S_n with $n = 1$ and the star graph S_m with $m = 2$, so we can give the double star graph $S_{1,2}$ represented as Figure 8.

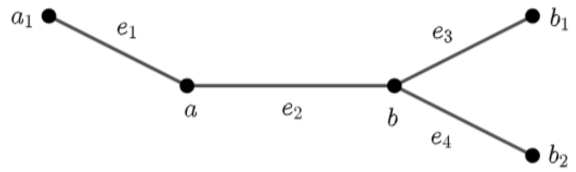

Figure 8. Double star graph $S_{1,2}$

Based on Figure 8 the double star graph $S_{1,2} \cong B_{2,2}$ consequently $|I(S_{1,2})| = |I(B_{2,2})|$ so the Fibonacci number of the double star graph $S_{1,2}$ is $i(S_{1,2}) = 2^{2+1} + 1 + 2^2 + 1$=14.
By applying the same method to find each $I(S_{1,m})$ the Fibonacci number of the double star graph $S_{1,m}$ is obtained, $i(S_{1,m}) = i(B_{2,m}) = i(B_{1,m}) + i(B_{0,m}) = 2^{m+1} + 1 + 2^m + 1 = 3(2^m + 1) - 1$.

3.3.3 Double Star Graph $S_{2,m}$

The double star graph $S_{2,2}$ is the graph constructed by the star graph S_n with $n = 2$ and the star graph S_m with $m = 2$, so we can give the double star graph $S_{2,2}$ represented as Figure 9.

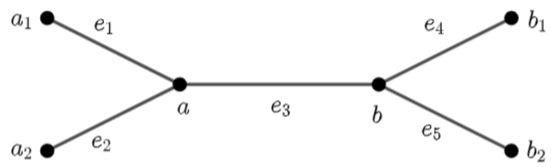

Figure 9. Double star graph $S_{2,2}$

Based on Figure 9 the collection of all independent sets on the double star graph $S_{2,2}$ is $I(S_{2,2}) = I(S_{1,2}) \cup \{\{a_2\}, \{a_2, b\}, \{a_2, b_1\}, \{a_2, b_2\}, \{a_2, b_1, b_2\}, \{a_1, a_2\}, \{a_1, a_2, b\}, \{a_1, a_2, b_1\}, \{a_1, a_2, b_2\}, \{a_1, a_2, b_1, b_2\}\}$ so that many collections of all sets $I(S_{2,2})$ can be obtained

$$
\begin{aligned}
|I(S_{2,2})| &= |I(S_{1,2})| + (C(2,0) + C(2,1) + 1 + C(2,2)) + (C(2,0) + C(2,1) + 1 + C(2,2)) \\
&= 14 + 1 + 3 + 1 + 1 + 3 + 1 \\
&= 24.
\end{aligned}
$$

By applying the same method to find each $I(S_{2,m})$ of the Fibonacci numbers of the double star graph $S_{2,m}$ is obtained. Let $X_2 = \{a_1\}$ and let X_2' denote the set whose elements are obtained by combining each subset of X_2 with $\{a_2\}$. Consequently, the cardinality of X_2' satisfies $|X_2'| = 2^{|X_2|}$.

$$
\begin{aligned}
|I(S_{2,m})| &= |I(S_{1,m})| + |\{a_2\} \cup I(S_{0,m})| + |\{a_2, a_1\} \cup I(S_{0,m})| \\
&= |I(S_{1,m})| + |X'||I(S_{0,m})| \\
&= |I(S_{1,m})| + 2^{|X_2|}|I(S_{0,m})| \\
&= i(S_{1,m}) + 2i(S_{0,m}).
\end{aligned}
$$

From the formula $|I(S_{2,m})|$ it can be found that the Fibonacci number of the double star graph $S_{2,m}$ is

$$i(S_{2,m}) = i(S_{1,m}) + 2i(S_{0,m}).$$

In the same way to find each $|I(S_{n,m})|$ the Fibonacci numbers of the double star graph $S_{n,m}$ are presented in Table 2.

Table 2. Formula $|I(S_{n,m})|$ with $n \geq 1$ and $m \geq 1$

No	Type of Graph	$i(S_{n,m})$
1.	$S_{0,m}$	$i(S_m)$
2.	$S_{1,m}$	$3 \times i(S_{0,m}) - 1$
3.	$S_{2,m}$	$i(S_{1,m}) + 2 \times i(S_{0,m})$
4.	$S_{3,m}$	$i(S_{2,m}) + 2^2 \times i(S_{0,m})$
⋮	⋮	⋮
n.	$S_{n,m}$	$i(S_{n-1,m}) + 2^{n-1} \times i(S_{\mathbf{0},m})$

Theorem 4 Let $S_{n,m}$ be a double star graph, then $i(S_{n,m}) = i(S_{n-1,m}) + 2^{n-1} \times i(S_{0,m})$.
Proof. Based on Table 2, we obtain $i(S_{1,m}) = 3(2^m) - 1$. Let $N_b(S_{n,m})$ denote the set of all independent sets of the double star graph $S_{n,m}$ written in binary form. This set can be partitioned into n subsets, denoted by

$$N_0(S_{n,m}), N_1, \dots, N_{n-1}$$

where each subset is defined according to the pattern of leading digits in the binary representation. Consider the vertex set $V(S_{n-1,m}) \cup \{a_n\}$, where $(a_n, a) \in E(S_{n,m})$. For any $x \in N_0(S_{n,m})$, the first digit of x is 0, which indicates that a_n does not appear in the independent set. Removing this initial digit yields a shorter binary string $x' \in N_b(S_{n-1,m})$. Conversely, appending a 0 to each string in $N_b(S_{n-1,m})$ generates all members of $N_0(S_{n,m})$. Therefore,

$$\mid N_0(S_{n,m}) \mid = i(S_{n-1,m}).$$

Next, define the subsets $N_0, N_1, \dots, N_j$ as follows, where each $*$ denotes an element of $\{0,1\}$:
$N_0 = \{00 \cdots 0 ** \cdots *\}$ with the first n digits equal to 0,
$N_1 = \{00 \cdots 01 ** \cdots *\}$ with the n-th digit equal to 1,
$N_2 = \{00 \cdots 011 ** \cdots *\}$ with the n-th and $(n-1)$-th digit equal to 1,
$\vdots$
$N_j = \{00 \cdots 011 \cdots 1 ** \cdots *\}$ with digits $(n - j + 1)$ through n equal to 1
Thus,

$$N_b(S_{n,m}) = N_0(S_{n,m}) \cup \bigcup_{i=1}^{n-1} N_i.$$

Furthermore,
$|N_0| = i(S_{0,m})$
$|N_1| = C((n-1), 0). i(S_{0,m})$
$|N_2| = C((n-1), 1). i(S_{0,m})$
$\vdots$
$|N_j| = C((n-1), (j-1)). i(S_{0,m})$.
Hence,

$$\left|\bigcup_{i=1}^{n-1} N_i\right| = \sum_{j=1}^{n-1} C((n-1), (j-1)). (i(S_{0,m})) = 2^{n-1}(i(S_{0,m}))$$

Combining these results, we arrive at the recurrence relation

$$i(S_{n,m}) = i(S_{n-1,m}) + 2^{n-1} \times i(S_{0,m}).$$

This completes the proof.
Corollary 2 Given $S_{n,m}$ be a double star graph, then

$$i(S_{0,m}) = 2^m + 1$$

and

$$i(S_{n,m}) = (2^n + 1)(2^m + 1) - 1$$

for $n, m \geq 1$.
Proof. Let $a_n = i(S_{n,m})$, then we have a recurrence relation, $a_n = a_{n-1} + 2^{n-1}a_0$.
Then the corresponding characteristic equation of

$$a_n - a_{n-1} = 0$$

is

$$r - 1 = 0.$$

Solving this equation yields,

$$r = 1.$$

Since the roots are different, the general homogeneous solution can be written as,

$$a_n^{(h)} = C.$$

The particular solution can be written as,

$$a_n^{(p)} = 2^n a_0$$
$$a_n = C + 2^n a_0.$$

Next, we use the initial values:

$$a_1 = 3(2^m) + 2.$$

Thus, we obtain the following system:

$$3(2^m) + 2 = C + 2a_0$$
$$C = 3(2^m) + 2 - 2a_0$$

Since $a_n = i(S_{n,m})$, we obtain

$$i(S_{n,m}) = \mathrm{C} + 2^{\mathrm{n}}(2^m + 1)$$

Therefore,

$$i(S_{n,m}) = (2^n + 1)(2^m + 1) - 1.$$

4. CONCLUSION

Through a detailed combinatorial analysis, this paper derives explicit recurrence relations together with closed-form expressions for the Fibonacci numbers associated with the broom graph $B_{n,m}$ and the double star graph $S_{n,m}$. It is shown that the Fibonacci number of $B_{n,m}$ satisfies a second order recurrence of Fibonacci type, while the corresponding sequence for $S_{n,m}$ is described by a recurrence that incorporates both linear and exponential terms. These findings clarify the role played by the underlying graph structure in shaping the growth of independent sets, highlighting the contrasting effects of path extension and star like branching. From a structural perspective, the recurrence relation for $B_{n,m}$ arises naturally from the incremental expansion of its path component and is closely related to decomposition principles known for path graphs. In contrast, the recurrence for $S_{n,m}$ reflects the combinatorial contributions of independent selections within the star component once the central vertex is specified. In addition to their theoretical interest, the explicit formulas obtained provide a practical tool for the efficient enumeration of independent configurations, with potential applications in areas such as network reliability and chemical graph theory. Moreover, the combinatorial approach developed here can be extended in a natural way to broader classes of broom like graphs with multiple attached paths, as well as to generalized double star graphs under additional structural constraints, thereby opening several avenues for further investigation of Fibonacci type invariants in graph theory.

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