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Bifurcation Analysis of Cubic Type Nonlinear Schrödinger Equation with Dispersion and Attenuation in Optical Fiber

[1] Rika Saputri
Mathematics Graduated Program, Universitas Syiah Kuala, Banda Aceh, Indonesia
[2] Marwan Ramli
Department of Mathematics, Universitas Syiah Kuala, Banda Aceh, Indonesia
[3] Muhammad Ikhwan
Department of Mathematics, Universitas Syiah Kuala, Banda Aceh, Indonesia
[4] Harish Abdillah Mardi
Graduate School of Mathematics and Applied Sciences, Universitas Syiah Kuala, Banda Aceh, Indonesia

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ABSTRACT

This study investigates the bifurcation of the cubic Nonlinear Schrödinger Equation (NLSE) in optical fiber media by considering dispersion and attenuation effects. The NLSE models light pulse propagation, with γ representing the strength of nonlinearity. Analytical derivations yield stationary solutions, while numerical simulations using the Newton–Raphson method and eigenvalue analysis verify stability. Results show a critical bifurcation at $\gamma=0$: for $\gamma \leq 0$, the system exhibits one unstable fixed point, whereas for $\gamma > 0$, three fixed points appear, with simulations confirming that only the two nontrivial branches are stable. This corresponds to a pitchfork bifurcation and a stability transition governed primarily by nonlinearity. Although attenuation is included in the model, its contribution is negligible, indicating that bifurcation behavior is dominated by γ. Compared with previous studies focusing on dispersion-only NLSE or fractional/extended models, this work highlights the decisive role of nonlinearity in determining fixed points and stability in optical fibers.

Corresponding Author:

Marwan Ramli
Department of Mathematics,
Universitas Syiah kuala, Banda Aceh, Indonesia
Email: marwan.math@usk.ac.id

1. INTRODUCTION

Partial differential equations (PDEs) are one of the equations that are widely used to observe natural phenomena. One of PDEs introduced by Erwin Schrödinger in 1925 is the Nonlinear Schrödinger Equation (NLSE) which developed from John Scott Russel's (1808-1882) observations of the water wave phenomenon in the Edinburgh-Glasgow channel [1-4]. The observed water waves did not change shape over a relatively long period of time along the channel. This phenomenon was introduced by Norman Zabusky and Martin Kruskal with the term soliton when studying solitary waves in the Korteweg-de Vries (KdV) equation [5]. The NLSE plays a fundamental role in modeling wave propagation in nonlinear media, particularly in optical fibers. It describes the evolution of complex wave envelopes and accounts for the effects of dispersion and nonlinearity. A crucial connection arises here when the dispersive spreading of a pulse is exactly counterbalanced by nonlinear self-phase modulation, the NLSE admits a soliton solution. These optical solitons are highly significant because they can travel long distances through fibers without distortion. The pioneering work of Hasegawa and Tappert [6] theoretically predicted soliton transmission in fibers, which was later confirmed experimentally by Mollenauer et al. [4]. This makes soliton theory directly applicable to fiber optics, providing the foundation for stable long-haul communication systems.

In this study, NLSE equation is investigated by incorporating dispersion and attenuation effects. Dispersion in optical fibers arises from differences in phase and group velocities among frequency components, resulting in pulse broadening and is mathematically modeled by the group dispersion coefficient β [7]. In contrast, attenuation describes the reduction of optical signal amplitude due to absorption and scattering in the medium, and is represented by the attenuation parameter α, indicating the rate of energy loss in the optical system [8]. The theoretical foundation of the Schrödinger equation, originally introduced by Erwin Schrödinger, is widely used due to its mathematically simpler and more practical formulation. Today, the Schrödinger equation has become a fundamental postulate of quantum mechanics [9].

Research on optical pulse propagation in fibers began with the seminal work of Hasegawa and Tappert [6], who introduced the NLSE to describe solitons in dispersive optical media. Since then, many studies have analyzed dispersion effects as a key factor in soliton formation and stability [10], [11],[12]. Later, research expanded to nonlinear effects, considering cubic, quintic, or fractional interactions, which led to complex dynamics such as multi-peak solitons, modulational instability, and bifurcation patterns [13], [14], [2],[1],[15]. Further bifurcation analyses were conducted in the presence of double-well potentials, coupled systems, and extended models such as the Fokas or Ginzburg–Landau equations [16], [3], [17], [18], [19], [20]. These developments established the NLSE as a fundamental framework in nonlinear optics and PDE bifurcation studies.

However, over long transmission distances, nonlinear phenomena in optical fibers begin to affect signal quality and system performance. One of the significant nonlinear phenomena is the interaction of optical signals with the transmission medium, which can give rise to phenomena such as soliton pulse formation, signal bending, and other effects related to dispersion and attenuation. To understand and manage these effects, mathematical models that describe the behavior of optical waves are essential. NLSE equation is the most commonly used model to describe the dynamics of pulses in optical fibers by considering dispersion, nonlinear effects, and attenuation [11]. However, complex nonlinear systems such as this can exhibit behavior that is very sensitive to initial conditions and system parameters, including the occurrence of bifurcations. Bifurcation in this context refers to a profound change in the structure of the solution of the NLSE equation when system parameters are changed [21]. Bifurcation analysis allows a deeper understanding of how small changes in parameters such as dispersion, nonlinear effects, and attenuation can affect pulse dynamics, which in turn can affect the performance of optical communication systems.

However, the role of attenuation in the cubic NLSE bifurcation framework has not been explicitly analyzed. Even recent works (Ramli et al. [1], Mardi et al. [2], Alotaibi et al. [3]) mainly emphasized dispersion and higher-order nonlinearities, without considering the simultaneous interaction of dispersion, nonlinearity, and attenuation in the basic cubic model. This study addresses that gap by analyzing the cubic NLSE with all three effects and demonstrates that the nonlinearity parameter γ dominates the bifurcation structure, while attenuation plays only a negligible role. By clarifying this interplay, our results contribute to a more complete understanding of how dispersion, nonlinearity, and attenuation shape the existence and stability of fixed points in optical fiber systems.

This research aims to analyze the bifurcation of the NLSE equation with dispersion and attenuation by applying analytical and numerical techniques. The study will reveal how changes in γ affect the number and type of fixed points and will identify the bifurcation patterns that emerge. The findings are expected to contribute to the mathematical modeling of optical fibers and enhance the design of robust optical communication systems.

2. RESEARCH METHOD

2.1 Nonlinear Schrödinger Equation (NLSE)

NLSE is an electromagnetic wave equation that describes the complex amplitude of the electric field in the direction of propagation. NLSE equation is also known as a mathematical model of nonlinear optical pulses that has been widely studied since the last century.

The NLSE equation takes into account effects such as dispersion, nonlinearity, and attenuation, using a NLSE as follows[2].

$$i\frac{\partial\Psi}{\partial t}+\frac{\beta}{2}\left(\frac{\partial^2\Psi}{\partial x^2}\right)+\gamma|\Psi|^2\Psi=\mathrm{V(x)}\Psi+i\frac{\alpha}{2}\Psi \tag{1}$$

where:

$\Psi(x,t)$: complex function representing the wave envelope of a light pulse in an optical fiber
β : dispersion parameter
γ : nonlinearity parameter related to the Kerr effect
$V(x)$: external potential modeling the trapping effect
α : attenuation parameter
i : imaginary number
x : spatial coordinate (position along the fiber)
t : pulse propagation time in the laboratory frame.

2.2 Transformation of Nonlinear Schrödinger Equation

In many studies of NLSE equations, the soliton solution is not directly sought from the general form, because the equations are nonlinear and complex. An ansatz is an assumption of the form of the solution of a partial differential equation. This ansatz is commonly used to find stationary, localized, and non-propagating solitons [22]. Ansatz to derive stationary solutions that use for soliton-type solution is [2]

$$\Psi(x,t) = \psi(x)e^{i\Omega t} \tag{2}$$

where $\psi(x)$ is real-valued. This ansatz assumes that the soliton profile is stationary in time apart from a harmonic phase factor. Furthermore, we impose localized boundary conditions consistent with solitary waves, namely

$$\lim_{|x|\to\infty} \psi(x) = 0$$

or equivalently homogeneous Dirichlet conditions $\psi(\pm L) = 0$ in the numerical simulations. furthermore, it is stated that equation (1) as an ordinary differential equation in the following form

$$F(\psi) = -\Omega\psi + \frac{\beta}{2}\frac{d^2\psi}{dx^2} + \gamma|\psi|^2\psi - V(x)\psi - i\frac{\alpha}{2}\psi \tag{3}$$

2.3 Perturbation Theory and Stability Analysis

Perturbation technique is one of the analysis methods to determine the approximate solution of a nonlinear equation. This technique is useful for demonstrating, predicting, and describing phenomena in vibrating systems caused by nonlinear effects. This theory can also be applied to linear and nonlinear systems with variable coefficients with complex boundary conditions where the exact closed-form solution is unknown [21].

This stability analysis will be carried out using perturbation theory. In the ansatz, a disturbance term will be added so that it becomes

$$\Psi(x,t) = \big(\psi(x) + \varepsilon\varphi(x,t) + i\varepsilon\phi(x,t) + O(\varepsilon)\big)e^{(i\Omega t)} \tag{4}$$

with $0 < \varepsilon \ll 1$. Substituting this into the NLSE yields a linearized system at order ε, which can be formulated as an eigenvalue problem. The stability of the stationary solution is determined by the sign of the real part of the eigenvalues. If any eigenvalue has a positive real part, the solution becomes unstable [5].

2.4 Newton Raphson Method

The basis of the Newton-Raphson (NR) method is the Taylor series for a function with two or more variables. for solving the nonlinear algebraic system obtained from the discretization of the stationary NLSE, we employ the Newton–Raphson (NR) method. This choice is motivated by its quadratic convergence rate and robustness in handling strongly nonlinear equations, which are common in NLSE bifurcation problems. Compared with simple fixed-point iterations or gradient-based solvers, NR requires fewer iterations to reach convergence and provides higher accuracy near the solution[23]. Many other researchers have used NR method in studying differential equations [14][24]. The NR method can be written as follows:

$$\boldsymbol{x} = \boldsymbol{x}^{(0)} - \left(\left(\boldsymbol{J}(\boldsymbol{x}^{(0)})\right)^{-1} \boldsymbol{f}(\boldsymbol{x}^{(0)})\right) \text{ or } \boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} - \left(\left(\boldsymbol{J}(\boldsymbol{x}^{(n)})\right)^{-1} \boldsymbol{f}(\boldsymbol{x}^{(n)})\right)$$

This combination of analytical and numerical approaches allows for a deeper understanding of the dynamics of nonlinear wave propagation in optical fibers under different parameter conditions.

Table 1. Parameters and function used in the NLSE model simulation

Parameter and function	Description	Value
β	Dispersion coefficient	1
α	Attenuation coefficient	0.01
γ	Nonlinearity parameter	-1, 0, 1
$V(x)$	External potential	$0.5x^2$
$\psi(x, 0)$	Initial condition	$\sqrt{2} \cdot \text{sech}\,(\sqrt{2}x)$
Ω	Frequency shift	1

The stationary solution $\psi(x,t)$ is generally not obtainable analytically. Therefore, NR method is applied to solve the nonlinear discrete system resulting from numerical discretization. This method is chosen due to its quadratic convergence rate when provided with an appropriate initial guess [8]. In addition to accurately locating stationary roots, NR method is also employed to perform numerical simulations of the NLSE, thereby facilitating detailed bifurcation and stability analysis in the context of optical wave propagation. More information how to use NR as system can be seen in [1].

2.5 Bifurcation Analysis

Bifurcation analysis is conducted to examine changes in the number and stability of fixed points as system parameters, particularly γ, are varied. Bifurcations such as pitchfork bifurcations indicate transitions from symmetric to asymmetric solutions [25] . In the context of optical fibers, such bifurcations correspond to transitions between stable and unstable propagation modes induced by changes in nonlinearity strength.

The fixed point or equilibrium point of a system of equations is the simplest point. If the fixed point of a system is stable, then the system tends to persist or be stable even if there is a small disturbance. However, the system changes, resulting in the system's behavior which was initially fixed to oscillation due to changes in high or low parameters. Further changes in system parameters have been shown to result in more extreme changes in behavior, including higher periodicity, quasi-periodicity, and chaos [21]. Some types of bifurcations include saddle-node, transcritical, pitchfork, and Hopf bifurcation [26].

3. RESULT AND ANALYSIS

3.1 Analytical Solution of the Nonlinear Schrödinger Equation

The analytical solution of the cubic type NLSE equation with dispersion and attenuation is obtained by inserting the ansatz of equation (2) so that we obtain equation (3). The analytical solution will be sought by rewriting equation (3) in the following form:

$$F(\psi) = -\Omega\psi + \frac{\beta}{2}\left(\frac{d^2\psi}{dx^2}\right) + \gamma\psi^3 - V(x)\psi - i\frac{\alpha}{2}\psi = 0, \tag{5}$$

then the analytical solution is obtained as follows [6].

$$\Psi(x,t) = \sqrt{\frac{2\Omega}{\gamma}}\, sech\left(\sqrt{\frac{2\Omega}{\beta}}x\right)e^{i\Omega t}. \tag{6}$$

This solution based on recent research in NLSE dynamics [14].

3.2 Numerical Stationary Solution of the Nonlinear Schrödinger Equation

The numerical stationary solution of the NLSE equation will be constructed using the NR method which produces the soliton form. The form of the NLSE equation used to obtain this stationary soliton solution has been written in discrete form which can be expressed by the following equation,

$$-\Omega\psi_n + \frac{\beta}{2}(\psi_{n+1} - 2\psi + \psi_{n-1}) + \gamma\psi_n^3 - V(x)\psi_n - i\frac{\alpha}{2}\psi_n = 0 \tag{7}$$

where ψ_n is a discrete variation of a wave function. β is the dispersion parameter of wave propagation. γ is the nonlinearity of the material (nonlinearity) of the medium through which the propagating wave passes, V which is the trap potential and α is the attenuation of the discrete NLSE equation [29]. Equation (7) can be solved as stationary solution with zero attenuation. This condition will be fulfilled because this paper only check the stationary solution.

Numerical solution to obtain stationary soliton solutions will use the NR method for the nonlinear equation system as follows,

$$\boldsymbol{x}_{n+1} = x - \boldsymbol{J}^{-1}_{(2N+1)x(2N+1)}\boldsymbol{F}(\boldsymbol{x}) \tag{8}$$

where

$$\boldsymbol{x} = \begin{bmatrix} \psi_{-N} \\ \psi_{-N+1} \\ \vdots \\ \psi_0 \\ \vdots \\ \psi_{N-1} \\ \psi_N \end{bmatrix} \tag{9}$$

$\boldsymbol{x}$ is a matrix that represents the group wave variables $(\psi_{-N})$with a discrete domain $n \in \{-N, -N+1, \dots, N-1\}$ with $N \in \mathbb{Z}^+$ and $\psi_{-N-1} = \psi_{N+1} = 0$.

$$\boldsymbol{F}(\mathbf{x}) = \begin{bmatrix} F(\psi_{-N)} \\ F(\psi_{-N+1)} \\ \vdots \\ F(\psi_0) \\ \vdots \\ F(\psi_{N-1}) \\ F(\psi_N) \end{bmatrix}$$

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} -\Omega\psi_{-N} + \frac{\beta}{2}(\psi_{-N+1} - 2\psi_{-N}) + \gamma{\psi_N}^3 - \mathrm{V}(x)\psi_{-N} \\ -\Omega\psi_{-N+1} + \frac{\beta}{2}(\psi_{-N+2} - 2\psi_{-N+1} + \psi_{-N}) + \gamma{\psi_{-N+1}}^3 - \mathrm{V}(x)\psi_{-N+1} \\ \vdots \\ -\Omega\psi_0 + \frac{\beta}{2}(\psi_1 - 2\psi_0 + \psi_{-1}) + \gamma{\psi_0}^3 - \mathrm{V}(x)\psi_0 \\ \vdots \\ -\Omega\psi_{N-1} + \frac{\beta}{2}(\psi_N - 2\psi_{N-1} + \psi_{N-2}) + \gamma{\psi_{N-1}}^3 - \mathrm{V}(x)\psi_{N-1} \\ -\Omega\psi_N + \frac{\beta}{2}(-2\psi_N + \psi_{N-1}) + \gamma{\psi_N}^3 - \mathrm{V}(x)\psi_N \end{bmatrix} \tag{10}$$

F is the domain of the function ψ_n where $n \in \{-N, -N+1, \dots, N-1\}$ with $N \in \mathbb{Z}^+$ and $\psi_{-N-1} = \psi_{N+1} = 0$. F can be stated $F(\psi_N) = \big(F(\psi_{-N}),\ F(\psi_{-N+1}), \dots,\ F(\psi_{N-1}),\ F(\psi_N)\big)$.

Meanwhile, $\boldsymbol{J}^{-1}$ is the inverse Jacobian matrix of order (2N + 1) × (2N + 1) constructed from the formation of $F\ (\psi)$ in the NLSE equation. The constructed solution will be in the form of a distribution of values that can be in the form of a soliton. The use of variations in parameters will affect the soliton obtained. Therefore, the use of parameters varied in the study is dimensionless, so that each result listed in this study has no units [9].

3.3 Equilibrium Points of NLSE

Equation (3) can be transformed into two first-order complex ordinary differential equations of the form

$$\frac{d\psi}{dx} = \phi \tag{11}$$

$$\frac{d^2\psi}{dx^2} = \frac{d\phi}{dx} = \frac{2}{\beta}\left[\Omega\psi - \gamma\psi^3 + V(x)\psi + i\frac{\alpha}{2}\psi\right] \tag{12}$$

Thus, three equilibrium points are obtained from the cubic type NLSE equation with dispersion and attenuation as follows.

$$(\psi, \phi) \in \left\{\left(-\sqrt{\frac{\Omega + V(x)}{\gamma}}, 0\right), (0,0), \left(\sqrt{\frac{\Omega + V(x)}{\gamma}}, 0\right)\right\}$$

then the eigenvalues can be obtained in the following way:

$$\lambda_{1,2} = \pm\sqrt{\frac{2}{\beta}\left(\Omega - 3\gamma\psi^2 + V(x)\right)} \tag{13}$$

For point (0,0) obtained $\lambda_{1,2} = \pm\sqrt{\frac{2}{\beta}\left(\Omega + V(x)\right)}$ and for points $\left(-\sqrt{\frac{\Omega+V(x)}{\gamma}}, 0\right)$ and $\left(\sqrt{\frac{\Omega+V(x)}{\gamma}}, 0\right)$ obtained $\lambda_{1,2} = \pm 2i\sqrt{\frac{(\Omega+V(x))}{\beta}}$. Based on the calculation of eigenvalues around the equilibrium point, it is known that the stability of the system is greatly influenced by the parameter values Ω, V(x), α, and β. When the four parameters are positive, the real part of all eigenvalues is negative, so the system is in a stable condition at equilibrium points $\left(-\sqrt{\frac{\Omega+V(x)}{\gamma}}, 0\right)$ and $\left(\sqrt{\frac{\Omega+V(x)}{\gamma}}, 0\right)$.

3.4 Linear Stability of Nonlinear Schrödinger Equation

The stability analysis of the cubic type NLSE equation with dispersion and attenuation is carried out using perturbation theory, namely adding a disturbance term as in equation (4) where $\varepsilon \in R$ is a constant with a very small value $(0 < \varepsilon \ll 1)$ then $\varphi(x,t)$ and $\phi(x,t)$ are real-valued functions F. After inserting the disturbance ansatz equation (4) into equation (1) we obtain

$$\begin{aligned}\left(i\varepsilon\frac{\partial\varphi}{\partial t} - \varepsilon\Omega\varphi - \varepsilon\frac{\partial\phi}{\partial t} - i\varepsilon\Omega\phi\right) &= -\frac{\beta}{2}(\varepsilon\varphi'' + i\varepsilon\phi'') - (3\gamma\varepsilon\varphi\psi^2 + i\gamma\varepsilon\phi\psi^2) + (\varepsilon V(x)\varphi + i\varepsilon V(x)\phi) \\ &\quad - \left(i\varepsilon\frac{\alpha}{2}\varphi - \varepsilon\frac{\alpha}{2}\phi\right)\end{aligned} \tag{14}$$

By separating equation (13) into real and imaginary parts, then the NLSE is obtained which can be written in the form

$$\lambda\phi = \left(\frac{\beta}{2}\frac{\partial^2}{\partial x^2} - \Omega + 3\gamma\psi^2 - V(x)\right)\varphi - \frac{\alpha}{2}\phi \tag{15}$$

$$\lambda\varphi = \left(-\frac{\beta}{2}\frac{\partial^2}{\partial x^2} - \gamma\psi^2 + V(x) + \Omega\right)\phi - \frac{\alpha}{2}\varphi \tag{16}$$

Furthermore, stability is obtained when $\alpha \approx 0$ (as shown in stationary solution), so that a matrix can be formed as follows.

$$\lambda\begin{bmatrix}\phi \\ \varphi\end{bmatrix} = \begin{bmatrix} 0 & \frac{\beta}{2}\frac{\partial^2}{\partial x^2} - \Omega + 3\gamma\psi^2 - V(x) \\ -\frac{\beta}{2}\frac{\partial^2}{\partial x^2} - \gamma\psi^2 + V(x) + \Omega & 0 \end{bmatrix}\begin{bmatrix}\phi \\ \varphi\end{bmatrix} \tag{17}$$

Then the distribution of real eigenvalues $(\lambda\varphi)$ and the distribution of imaginary eigenvalues $(\lambda\phi)$ for the NLSE equation will be obtained so that the stability of the equation can be analyzed [1].

3.5 Bifurcation Analysis of Nonlinear Schrödinger Equation

This study based on the fixed points and eigenvalues obtained in sub-chapter 3.3, it is known that when the parameter $\gamma < 0$ and $\gamma = 0$, the stationary solution $(\psi, \phi) = (0,0)$ is obtained, there is only 1 fixed point and it is unstable because physically when $\gamma < 0$, a dark soliton will be formed, while when $\gamma = 0$, the solution cannot be defined. Furthermore, when $\gamma > 0$, there are three fixed points obtained, namely $\left(-\sqrt{\frac{\Omega+V(x)}{\gamma}}, 0\right)$, $(0,0)$, and $\left(\sqrt{\frac{\Omega+V(x)}{\gamma}}, 0\right)$. The stability around the fixed point $(\psi, \phi) = (0,0)$ is unstable and the stability around the fixed point $\left(-\sqrt{\frac{\Omega+V(x)}{\gamma}}, 0\right)and\left(\sqrt{\frac{\Omega+V(x)}{\gamma}}, 0\right)$based on the explanation of the eigenvalues in sub-chapter 3.3 is stable with the condition that the parameters Ω, V(x), α, and β are positive. So that the bifurcation of the cubic type NLSE equation with dispersion and attenuation can be seen in Figure 1 below.

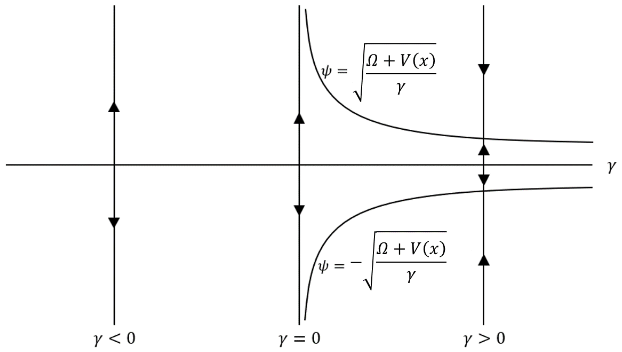


Figure 1. Bifurcation diagram of the NLSE equation illustrating pitchfork bifurcation behavior.

Figure 1 shows that the horizontal axis is the γ parameter axis and the vertical axis is the ψ axis. There is a separator between the $\gamma < 0, \gamma = 0$, and $\gamma > 0$ regions. When $\gamma < 0$, the arrow moves away from the point $(0,0)$ indicating that the stability around the point $(0,0)$ is unstable. When γ=0, the solution becomes undefined because there is a division by zero. This condition is causing the arrow to also move away from the equilibrium point $(0,0)$. Then there is a change in the behavior of the ψ solution, namely when $\gamma > 0$ two branches appear that are symmetric to $\psi = 0$, namely $\psi = -\sqrt{\frac{\Omega+V(x)}{\gamma}}$ and $\psi = \sqrt{\frac{\Omega+V(x)}{\gamma}}$ and based on the arrows in Figure 1 it shows that the two new points that appear are stable. It can be seen that when γ is varied, there is a change in the number of fixed points, namely when $\gamma < 0$ and $\gamma = 0$ there is 1 fixed point, when $\gamma > 0$ there are 3 fixed points. The point $\gamma = 0$ is a critical point because there is a change in the behavior of the solution. So that Figure 1 is called a Pitchfork bifurcation, namely a solution that branches from one trivial solution to two non-trivial solutions when γ passes the critical point.

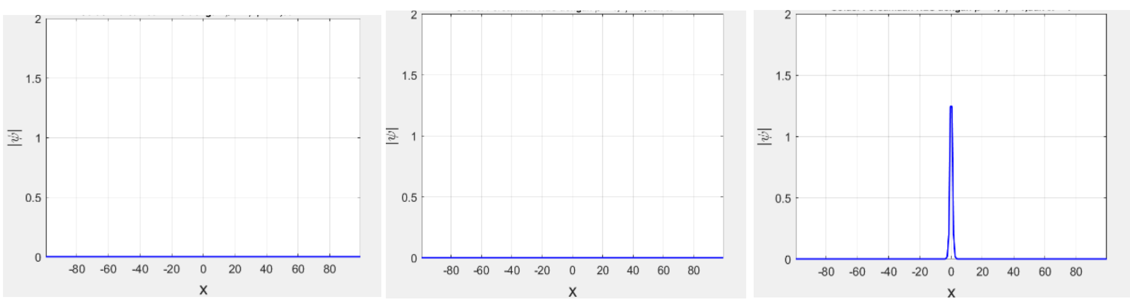
The effect of the nonlinear parameter γ on the number of fixed points can be explained more explicitly. When $\gamma < 0$, the stationary solution is given by $(\psi, \phi) = (0,0)$. This corresponds to a single fixed point which is unstable because the eigenvalues have a positive real part. Physically, this condition is associated with the emergence of a soliton structure where the amplitude of the wave decreases rather than forming a localized pulse.

At $\gamma = 0$, the stationary solution becomes undefined since the expression for the fixed points involves division by γ. Mathematically, this condition acts as a singularity, while physically it implies that no soliton solution can exist because the nonlinear contribution of the medium vanishes.

When $\gamma > 0$, three fixed points emerge: (0,0) which remains unstable, and two symmetric non-trivial fixed points, $\left(\pm\sqrt{\frac{\Omega+V(x)}{\gamma}}, 0\right)$, which are both stable. This transition represents the branching of solutions, where the trivial fixed point loses stability and two new symmetric stable states appear. The bifurcation at $\gamma = 0$ therefore marks a qualitative change in the system dynamics: from a single unstable equilibrium into three equilibria with two stable branches. This change illustrates the classical behavior of a supercritical pitchfork bifurcation in the cubic NLSE with dispersion and attenuation.

3.6 Simulation of Numerical Solution of Nonlinear Schrödinger Equation

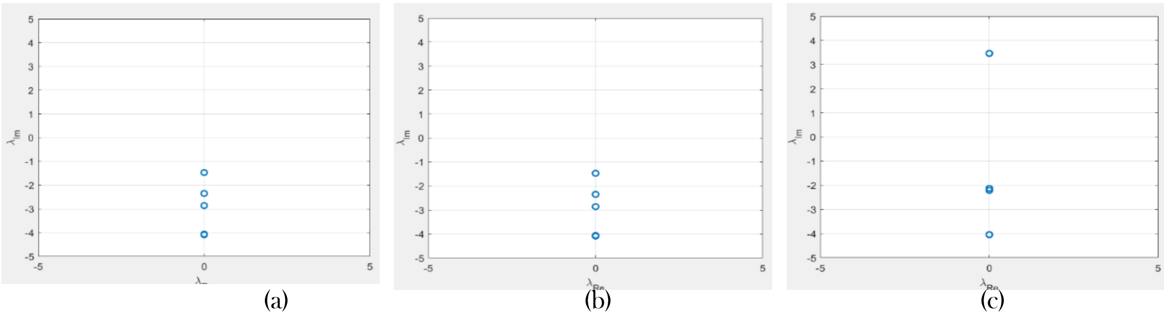
Numerical simulations were performed using MATLAB by entering parameters as in table 1 to obtain Figure 2 as follows:



(a) (b) (c)

Figure 2. Numerical solutions of the NLSE equation with potential $V(x) = 0{,}5x^2, \Omega = 1, \alpha = 0{,}01, \beta = 1$, (a) $\gamma = -1$, (b) $\gamma = 0$, and (c) $\gamma = 1$.

A significant change occurs when $\gamma = 1$. The simulation reveals three distinct fixed points: two symmetric stable points and one central unstable point. The phase portrait shows bifurcation behavior in which the system splits into multiple attractors. This phenomenon is characteristic of a pitchfork bifurcation, where increasing γ causes a qualitative shift in system dynamics. The appearance of multiple equilibria and symmetry breaking marks a critical transition in the system's stability structure.



(a) (b) (c)

Figure 3. Eigenvalue distributions of the NLSE system for $V(\mathrm{x}) = 0{,}5\mathrm{x}^2, \Omega = 1, \alpha = 0{,}01, \beta = 1,$ (a) $\gamma = -1$, (b) $\gamma = 0$, and (c) $\gamma = 1$.

The bifurcation diagrams and time-evolution plots support these findings, showing how solution trajectories vary significantly with small changes in the parameter γ. These results highlight the importance of nonlinearity in shaping wave behavior in optical fibers. Physically, stability implies that optical pulses can propagate over long distances while preserving their shape and energy, ensuring reliable signal transmission. Instability, in contrast, corresponds to pulse broadening, distortion, or even decay, which degrades transmission quality and limits the performance of optical communication systems. This suggests that careful control of system parameters is essential in designing stable optical transmission models.

Figure 2 further illustrates this bifurcation behavior through numerical simulations. In Figure 2(a), for $\gamma = -1$, only one equilibrium point appears and it is unstable, consistent with the analytical prediction of a single dark soliton state. The solution tends to diverge, showing that the system cannot sustain a localized pulse.

In Figure 2(b), for $\gamma = 0$, the solution is undefined, which is numerically observed as the disappearance of stable trajectories around the origin. This confirms that at the critical point no stationary soliton can exist.

In contrast, Figure 2(c) for $\gamma = 1$ shows the appearance of three equilibria: a central unstable fixed point at $\psi = 0$ and two symmetric stable equilibria at $\psi = \pm\sqrt{\frac{\Omega + V(x)}{\gamma}}$. These two new branches represent stable bright soliton states. The phase portrait indicates that trajectories are attracted to these stable states, confirming the supercritical pitchfork bifurcation scenario.

The same behaviour can be observed from the eigenvalue distributions in Figure 3. For $\gamma = -1$, the eigenvalues have positive real parts, confirming instability. At $\gamma = 0$, the eigenvalues approach the imaginary axis, representing the critical bifurcation point. For $\gamma = 1$, the eigenvalues corresponding to the two non-trivial equilibria have negative real parts, ensuring stability, while the central equilibrium remains unstable. This numerical evidence is consistent with the theoretical bifurcation analysis and highlights the role of γ as a bifurcation parameter that determines both the number and the stability of the fixed points in the NLSE system.

4. CONCLUSION

This study analyzed the bifurcation behavior of the cubic NLSE with dispersion and attenuation to model nonlinear wave propagation in optical fibers. Both analytical and numerical investigations demonstrate that variations in the nonlinearity parameter γ induce a supercritical pitchfork bifurcation. Specifically, when $\gamma \leq 0$ the system admits only one (unstable) fixed point, while for $\gamma > 0$ two additional symmetric stable equilibria emerge. This transition highlights how nonlinearity governs the stability and multiplicity of soliton solutions in optical systems.

From a practical perspective, stability corresponds to the ability of optical pulses to maintain their shape and energy during transmission, which is essential for reliable high-speed communication. Instability, by contrast, may result in pulse broadening, distortion, or decay, leading to signal degradation. These insights emphasize the importance of controlling nonlinear effects when designing stable optical transmission schemes.

This work is limited by its focus on the cubic NLSE with simple dispersion and attenuation terms, neglecting higher-order dispersion, stochastic noise, and gain–loss mechanisms that occur in real fibers. Future research should therefore examine more complex trapping potentials, higher-order and fractional dispersion, nonlinear loss or gain, and the role of negative nonlinearity. In addition, extending the analysis to time-dependent perturbations and validating the model against experimental or simulation data would strengthen its relevance to practical optical communication systems.

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