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CHARACTERISTICS OF SHACKLE GRAPH: $\boldsymbol{Shack(K_n, v_{(j,i)}, t)}$, $\boldsymbol{Shack(S_n, v_{(j,i)}, t)}$, & $\boldsymbol{Shack(K_{(n,n)}, v_{(r_j,1)}, t)}$

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Abstrak:

Operasi schackle adalah operasi antara dua atau lebih graf yang menghasilkan graf baru. Graf shackle dinotasikan $shackle\ (G_1, G_2, \dots, G_t)$ adalah graf yang dihasilkan dari t salinan dari graf G yang diberi simbol dengan $Shack(G, t)$ dimana $t \geq 2$ dan t bilangan asli. Operasi shackle ppada penelitian ini adalah shackle titik. Operasi shackle titik dinotasikan dengan $shackle\ (G, v, t)$ artinya graf yang dibangun dari sembarang graf G sebanyak t salinan dan titik v sebagai $linkage\ vertex$. Kelas graf yang akan di eksporasi karakterisinya dan bilangan kromatinya adalah $Shack(K_n, v_{(j,i)}, t)$, $Shack(S_n, v_{(j,i)}, t)$, & $Shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)$. Hasil penelitiannya menunjukkan bahwa bilangan kromatik graf shackle sama dengan subgraf pembangunnya.

Kata Kunci:

Operasi Shackle, Shackle titik, graf shackle, bilangan kromatik.

Abstract:

A shackle operation is an operation between two or more graphs that results in a new graph. Shackle graph notated $Shack(G_1, G_2, \dots, G_t)$ is a product graph from t copy of graph G is denoted by $Shack\ (G, t)$ where $t \geq 2$ and t are natural numbers. The shackle operation in this research is vertex shackle. Vertex shackle operation is denoted by $Shack\ (G, v, t)$ which means that the graph is constructed from any graph G as many as t copies and vertex v as linkage vertex. The class of graphs examined in this study are $Shack(K_n, v_{(j,i)}, t)$, $Shack(S_n, v_{(j,i)}, t)$, & $Shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)$. The results show that the chromatic number of the shackle graph is the same as the subgraph that generates it.

***Keywords**:*

Shackle Operation, Vertex Shackle, Shackle Graph, Chromatic Numbers.

A. Introduction

Graph theory is an interesting study in Discrete Mathematics, which discusses the properties of graphs (Mujib, 2011, Diana, Suryaningtyas, & Suprapti, 2016). Graphs have long been known and are widely applied in various fields of science, which are still being developed now. Graphs have been widely used in computing, modeling, and even gaming (Diana et al., 2016, Maarif, 2017, Mujib & Assiyatun, 2011). In general, a graph is a set of pairs (V, E) where V represents a non-empty set of vertices and E is a set of edges connecting a pair of vertices on the graph. A Graph G is a pair of sets (V, E) where V is a non-empty set called vertices and $E \subseteq V^2$ which is a 2-element subset of V called sides (Bondy & Murty, 2008). The graph is presented in the form of a graph with the elements of the set V represented as points, while the elements of the set E are the lines connecting two corresponding vertices.

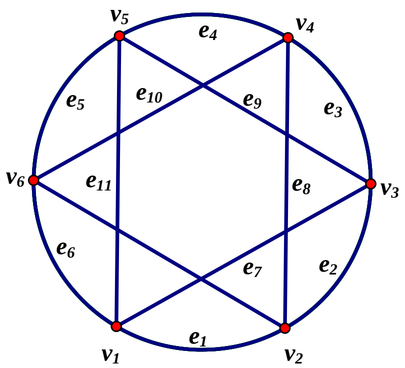

Figure 1. Graph $\boldsymbol{G = (V, E)}$

Figure 1 above, is an example of the graph $G = (V, E)$. Where is the set of vertices $V = \{v_i : i = 1,2,3, \ldots, 6\}$ and the set of sides $E = \{e_j : j = 1,2,3, \ldots, 11\}$. Side $e_2 = v_2v_3$ because it connects vertex v_2 with v_3. Therefore, it is called v_2 adjacent to v_3, besides v_2 is adjacent to vertices v_1, v_4, and v_6, while the relationship between v_2 and v_5 is not neighboring or is called independent. Let e_1 and e_2 are two edges of graph G. The edges e_1 and e_2 are called independent if e_1 and e_2 are not adjacent. Likewise, two vertices on G are independent of each other if they are not adjacent. Graph G is called a complete graph, denoted by K_n if every two vertices in G are adjacent. A graph G is called finite if the set of vertices $V(G)$ is finite. The number of vertices on graph G is called the order of G, denoted by $|V(G)|$, while the number of edges on graph G is called the size of G, denoted by $|E(G)|$. The number of edges attached to a vertex v in G is called the degree of the point v, denoted by $d_G(v)$ (Hartsfield & Ringel, 2003).

The smallest degree of graph G is denoted by $\delta(G)$, while the largest degree on graph G is denoted by $\Delta(G)$. Graph G in Figure 1 has a sequence of degrees $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_6) = 4$, while $\delta\ (G) = \Delta\ (G) = 4$. A graph G is called a regular graph if the degrees of the vertices are equal. A graph $G = (V, E)$ is called a $k - regular$ graph if $d_G(v) = k$ for every $v \in V$. The graph K_n is a graph $(n - 1) - reguler$, because each point has a degree of $n - 1$. The cycle graph of order n; $n \geq 3$ denoted S_n is a $2 - regular$ connected graph (Bondy & Murty, 2008).

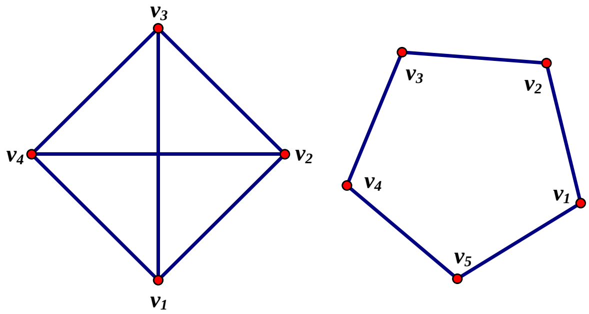

Figure 2. Graph $\boldsymbol{K_4}$ and Graph $\boldsymbol{S_5}$

Figure 2 shows a complete graph K_4, where each vertex has a degree of 3. Therefore, it is called a $3-regular$ graph. Whereas, S_5 is a 5-vertices cycle graph with each vertex having a degree of two, called $2-regular$. A graph $G=(V,E)$ is called a bipartite graph if V can be partitioned into two non-empty subsets X and Y such that for each edge of G one end is at X and the other is on Y. Suppose $|X| = m$ and $|Y| = n$ where $m,n \geq 2, m,n \in \mathbb{N}$. If every vertex in X is adjacent to a vertex in Y, then G is called a complete bipartite graph, denoted by $K_{m,n}$.

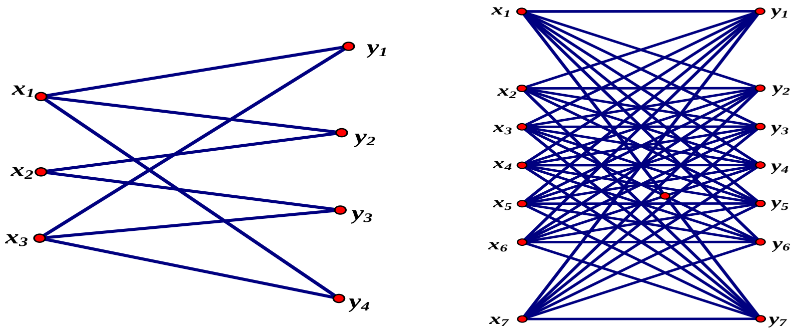

Figure 3. Graph $K_{3,4}$ and Graph $K_{7,7}$

The $K_{3,4}$ graph in Figure 3, is a bipartite graph but it is not complete. Because not every vertex in X connects at Y. Whereas Graph $K_{7,7}$ is a complete bipartite graph (Saifudin, 2020). In the study of graph theory, the operation between two graphs is one way to obtain new graphs. There are various types of operations on a graph, one of which is the shackle operation (Diestel, 2005). In this study, the researcher examines the characteristics of the graph $Shack\left(K_n, v_{(j,i)}, t\right)$, $Shack\left(S_n, v_{(j,i)}, t\right)$, and $Shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)$.

B. Theoretical Studies

Definition 1 (Maryati, Salman, Baskoro, Ryan, & Miller, 2010)

Let $k \geq 2$ be an integer. Define $shackle$ as a graph construction by connected non-trivial graphs $G_1, G_2, G_3, \dots, G_k$ such that G_s and G_t do not have a common vertex for every $s,t \in [1,k]$ with $|s-t| \geq 2$ and every $i \in [1, k-1]$ G_i dan G_{i+1} have exactly one common point called $linkage\ vertex$, and $k-1$ $linkage\ vertex$ is different. The shackle graph is denoted by $Shack(G_1, G_2, \dots, G_k)$.

Henceforth, the shackle graph is denoted by $shack(G, v, t)$. The graph $shack(G, v, t)$ means that the graph G is copied t times with vertex v as the vertex linkage. For example, Cycle Graph S_4 with $t = 3$ and vertex $x_i, i = 2,3$ as vertex linkage, then Figure 4 below is the result of the shackle operation on graph S_4.

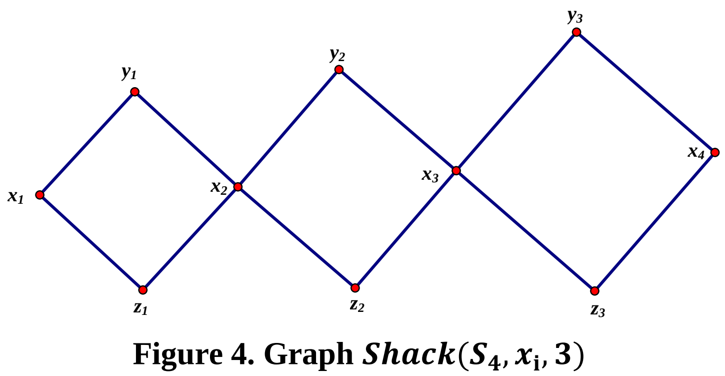

Figure 4. Graph $Shack(S_4, x_i, 3)$

Based on Figure 4, Graph $Shack(S_4, x_\mathrm{i}, 3)$ has a set of vertices, namely $V = \{x_i : i = 1,2,3,4\} \cup \{y_j : j = 1,2,3\} \cup \{z_k : k = 1,2,3\}$ and a set of edges, namely $E = \{x_i y_j, x_i z_i : i = j, i = 1,2,3\} \cup \{y_j x_i, z_k x_i : i = j + 1 \; or \; i = k + 1, j = k = 1,2,3\}$. The cardinalities of V and E respectively $|V| = 3|V(S_4)| - 2 = 10$ and $|E| = 3|E(S_4)| = 12$. In addition, the maximum degree of $Shack(S_4, x_\mathrm{i}, 3)$ is $\Delta(Shack(S_4, x_\mathrm{i}, 3)) = 4$ and the minimum degree of $Shack(S_4, x_\mathrm{i}, 3)$ is $\delta(Shack(S_4, x_\mathrm{i}, 3)) = 2$.

This article discusses a new graph of the product of the shackle operation. The graphs to be studied are $shack(K_n, v_i, t)$, $shack(S_n, v_i, t)$, and $shack(K_{n,n}, v_i, t)$. $Shack(K_n, v_i, t)$ is a complete graph K_n which is copied as many as t with the connecting vertex v_i where $i = 1,2,3, \dots, n$. Then, $shack(S_n, v_i, t)$ is a cycle graph (circle) with n vertices, which is copied by t with the connecting vertex v_i with $i = 1,2,3, \dots, n$. And $shack(K_{n,n}, v_i, t)$ is a complete graph.

C. Research methods

The method used in this research is qualitative research. The purpose of this study is to explore the properties of the graph shackle operation results. The following will be explained in more detail.

1. Types of research

This research uses a literature study approach. This is under the objectives of this study, that is to explore the characteristics and properties of the graph from the shackle operation. The graphs used are Complete graph (K_n), bipartite complete graph $(K_{n,n})$, and circle graph (S_n). In addition, this study aims to prove the chromatic number of graph $Shack(K_n, v_{(j,i)}, t)$, $Shack(S_n, v_{(j,i)}, t)$, and $Shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)$.

2. Research Subject

The subject of this research is a graph resulting from the shackle operation. Shackle graph is obtained from Complete graph (K_n), bipartite complete graph $(K_{n,n})$, and circle graph (S_n). More specifically, this study explores the characteristics and prove the chromatic number of graphs $Shack(K_n, v_{(j,i)}, t)$, $Shack(S_n, v_{(j,i)}, t)$, and $Shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)$.

3. Prosedure

This research procedure consists of four stages. First, it examines the definition of shackle operations on a graph. Based on the definition of the shackle operation, the researcher performs shackle operations on Complete graph (K_n), bipartite complete graph $(K_{n,n})$, and circle graph (S_n). Second, generalize the graphs $Shack(K_n, v_{(j,i)}, t)$, $Shack(S_n, v_{(j,i)}, t)$, and $Shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)$. Third, the generalization construct graphs $Shack(K_n, v_{(j,i)}, t)$, $Shack(S_n, v_{(j,i)}, t)$, and $Shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)$. And finally, explore the properties of graphs $Shack(K_n, v_{(j,i)}, t)$, $Shack(S_n, v_{(j,i)}, t)$, and $Shack\left(K_{(n,n)}, v_{(r_j,1)}, t\right)$. Conjecture the chromatic number. And the final stage, prove the chromatic number of Graph A.

D. Research Results and Discussion

1. Graph $\boldsymbol{Shack(K_n, v_i, t)}$

Based on Figure 5, without lost generalization, for example, we select vertex $v_{(1,i)}$ which is opposite to vertex $v_{(1,1)}$, then Graph $shack(K_n, v_{(j,i)}, t)$ is a graph that has a set of vertices $V = \{v_{(j,k)} : 1 \le j \le t,\; 1 \le k \le n, k \ne i\} \cup \{v_{(j,k)} : 1 \le j \le t, k = i\}$ and set of edges $E = \{e_{(1,k,l)} = (v_{(1,k)} v_{(1,l \ne k)}) : 1 \le k \le n, 1 \le l \le n\} \cup \{e_{(j,k,l)} = (v_{(j,k)} v_{(j,l \ne k)}) : 2 \le j \le t, 1 \le k, l \le n\} \cup \{e_{(j+1,i,l)} = (v_{(j,i)} v_{(j+1,l \ne i)}) : 1 \le j \le t - 1,\; 2 \le l \le n\}$. Where $|V| = (n - 1)t -$

1 and $|E| = tC_2^{n+1}$. Additionally obtained $\Delta\left(shack\left(K_n, v_{(j,i)}, t\right)\right) = 2(n-1)$ and $\delta\left(shack\left(K_n, v_{(j,i)}, t\right)\right) = n-1$.

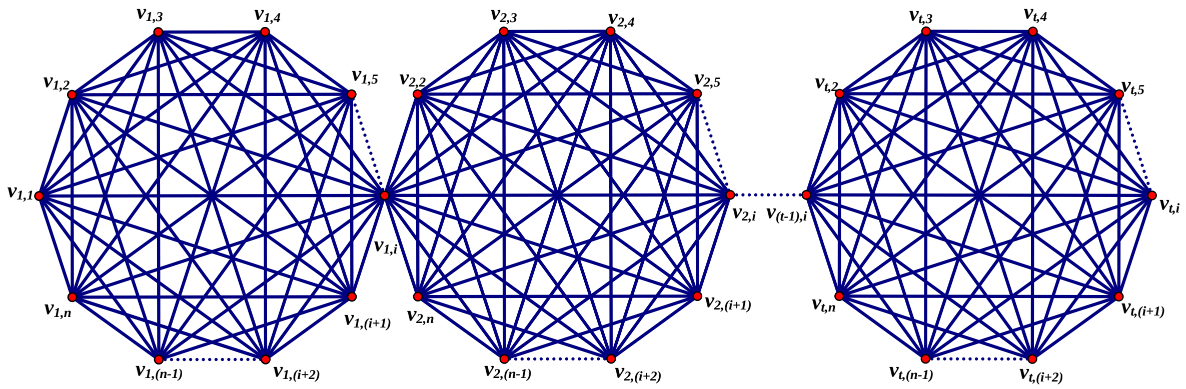

Figure 5. Graph $\boldsymbol{Shack(K_n, v_i, t)}$

Teorema 1.

The chromatic number of graphs $shack\left(K_n, v_{(j,i)}, t\right)$ is n, for every $n \in \mathbb{N}$.

$$\chi\left(shack\left(K_n, v_{(j,i)}, t\right)\right) = n, n \in \mathbb{N}$$

Proof.

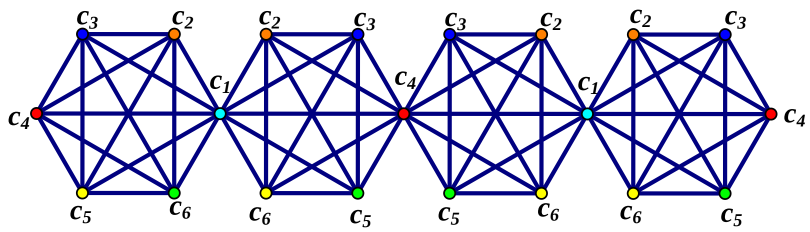

Figure 6. $\boldsymbol{\chi\left(shack\left(K_6, v_{(1,1)}, 4\right)\right) = 6}$

Graph $shack\left(K_n, v_{(j,i)}, t\right)$ has t subgraph K_n. Therefore, the chromatic number of the Graph $shack\left(K_n, v_{(j,i)}, t\right)$ depends on subgraph K_n. Subgraph K_n is a complete graph, each vertex is adjacent to another vertex, $d(v_i) = n-1$. Therefore, at least with the color set $C = \{c_1, c_2, c_3, \dots, c_n\}$, subgraph K_n can be colored. Thus, $\chi(K_n) = n$ (Mujib, 2011).

Next, we will show $\chi\left(shack\left(K_n, v_{(j,i)}, t\right)\right) = n, n \in \mathbb{N}$. Figure 6 as an illustration of shackle graph coloring. First, we do the coloring on subgraph 1, subgraph K_n with color set C, by coloring it counterclockwise. Furthermore, the coloring of subgraph 2 is colored clockwise with the color set C. And so on until subgraph t. So that $shack\left(K_n, v_{(j,i)}, t\right)$ can be colored with the color set C. Thus, $\chi\left(shack\left(K_n, v_{(j,i)}, t\right)\right) = n, n \in \mathbb{N}$ ■

2. Graph $Shack(S_n, v_i, t)$

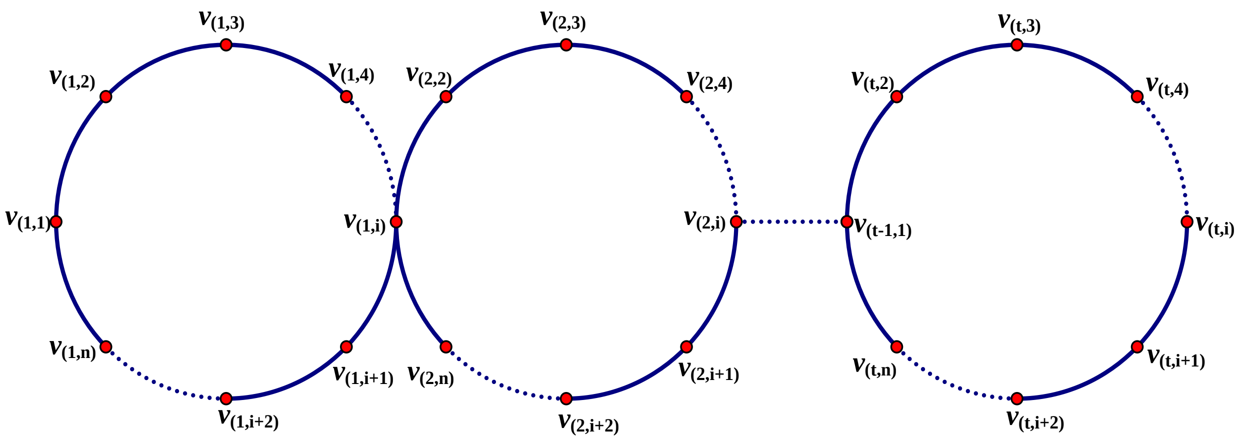

Figure 7. Graph $Shack(S_n, v_i, t)$

Figure 7 shows the construction of a $shack(S_n, v_{(j,i)}, t)$. Graph $shack(S_n, v_{(j,i)}, t)$ has the set of vertices $V = \left\{\{v_{(j,k)}: 1 \le j \le t,\ 1 \le k \le n, k \ne i\} \cup \{v_{(j,k)}: 1 \le j \le t, k = i\}\right\}$ dan the set edges $E = \{e_{(j,k)} = (v_{(j,k)}v_{(j,k+1)}): 1 \le j \le t, 1 \le k \le n\ \} \cup \{e_{(j+1,k)} = (v_{(j,i)}v_{(j+1,k)}): 1 \le j \le t, k = 2, n\ \}$. Obtained $|V| = (n-1)t - 1$, $|E| = tn$, $\Delta\left(shack(S_n, v_{(j,i)}, t)\right) = 4$, and $\delta\left(shack(S_n, v_{(j,i)}, t)\right) = 2$.

Teorema 2.
The chromatic number of graphs $shack(S_n, v_i, t)$ is 2 if n even or 3 if n odd, for every $n \ge 3, n \in N$.

$$\chi\big(shack(S_n, v_i, t)\big) = \begin{cases} 2, & if\ n\ even \\ 3, & if\ n\ odd \end{cases}$$

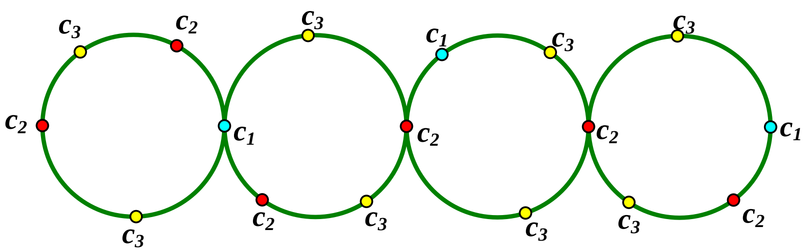

Figure 8. $\chi\big(shack(S_5, v_1, 4)\big) = 3$

Proof.
It is known that $\chi(S_n) = 2$ for n even and $\chi(S_n) = 3$ for n odd (Mujib, 2011). Because the shack graph $shack(S_n, v_i, t)$ has t subgraph S_n. Then the vertex coloring of the shack graph $shack(S_n, v_i, t)$ depends on the coloring of S_n. Vertex v_i as a link between subgraphs. Therefore, each subgraph can always be colored with the color in S_n. Thus, $\chi\big(shack(S_n, v_i, t)\big) = 2$ for n even, and $\chi\big(shack(S_n, v_i, t)\big) = 3$ for n odd.■

3. Graph $Shack\left(K_{n,n}, v_{(r_j,i)}, t\right)$

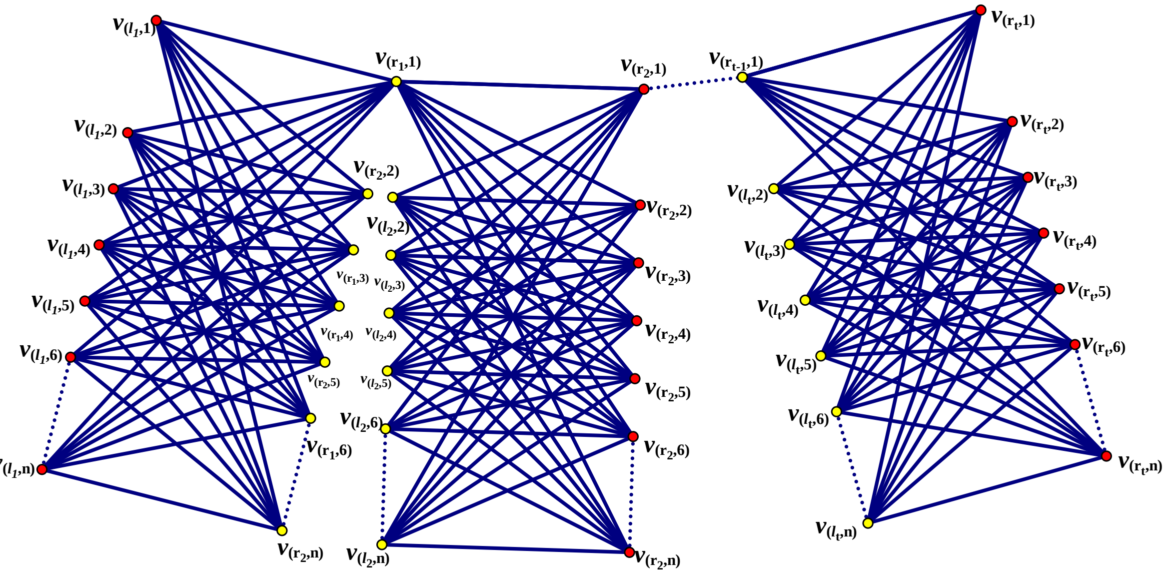

Figure 9. Graph $Shack\left(K_{n,n}, v_{(r_j,i)}, t\right)$

Graph $shack\left(K_{(n,n)}, v_{(r_j,i)}, t\right)$ is shown in Figure 9. Graph $shack\left(K_{(n,n)}, v_{(r_j,i)}, t\right)$ has the set of vertices $V = \left\{\{v_{(l_1,1)}\} \cup \left\{v_{(l_j,i)}: 1 \le j \le t,\ 2 \le i \le n\right\} \cup \left\{v_{(r_j,i)}: 1 \le j \le t, 1 \le i \le n\right\}\right\}$ and the set of edges $E = \left\{e_{(j,k)} = \left(v_{(l_j,k)}v_{(r_j,i)}\right): 1 \le j \le t, 1 \le k, i \le n\ \right\} \cup \left\{e_{(j,k)} = \left(v_{(l_j,k)}v_{(r_{j+1},i)}\right):\ 1 \le j \le t-1, 1 \le k, i \le n\ \right\}$. Obtained $|V| = 2nt$, $|E| = tn^2$, $\Delta\left(shack\left(K_{n,n}, v_{(j,i)}, t\right)\right) = 2n$, and $\delta\left(shack\left(K_{n,n}, v_{(j,i)}, t\right)\right) = n$.

Teorema 3.
The chromatic number of graphs$Shack\left(K_{n,n}, v_i, t\right)$ is, for every $n \in N$.

$$\chi\left(Shack\left(K_{n,n}, v_i, t\right)\right) = 2$$

Proof.
Graph $Shack\left(K_{n,n}, v_i, t\right)$ has t subgraph of $K_{n,n}$. Subgraph $K_{n,n}$ k-th is a bipartite subgraph consisting of two sets of vertices $V_{X_k} = \left\{v_{(l_k,i)}: 1 \le k \le t, 2 \le i \le n\right\} \cup \left\{v_{(r_{k-1},i)}: 2 \le k \le t\right\}$ and $V_{Y_k} = \left\{v_{(r_k,i)}: 1 \le k \le t, 1 \le i \le n\right\}$. therefore, $\chi\left(K_{n,n}\right) = 2$ (Mujib, 2011). Suppose given the color set $C = \{c_1, c_2\}$. Because every vertex in V_{X_k} independent, then with color c_1, the set vertices V_{X_k} is colored. In the same way for set V_{Y_k}, because every vertex in V_{Y_k} independent dan connected to the vertex on V_{X_k}, then it is enough to use color c_2. Since vertex $v_{(r_k,i)} \in V_{X_{k+1}} \cap V_{Y_k}$, subgraph $K_{n,n}$ $(k+1)th$, set of vertices $V_{X_{k+1}}$ colored with the same color as the vertex $v_{(r_k,i)}$ that is c_2. Meanwhile, the set of vertices V_{Y_k} will be colored c_1. Thus, $\chi\left(Shack\left(K_{n,n}, v_i, t\right)\right) = 2$ ■

E. Conclusion

Based on the results of the shackle operation on the complete graph K_n, the cycle graph S_n, and the Complete Bipartite graph $K_{n,n}$, several conclusions are obtained: 1) The $Shack\ (K_n, v_i, t)$ graph has a maximum degree of $2\ (n-1)$. This happens because a point is a linkage between components, so $\Delta\ (G)$ is a multiple of two. 2) The chromatic number of

$\chi\left(shack(K_n, v_{(j,i)}, t)\right) = \chi(K_n) = n$. 3) The $Shack(S_n, v_i, t)$ graph also has a maximum vertex degree multiple of two, so $\Delta(G) = 4$. 4) The chromatic number of $\chi(shack(S_n, v_i, t)) = \chi(S_n)$. 5) The $shack\ (K_{(n,n)}, v_{(j,i)}, t)$ graph is also the same, has $\Delta(G) = 2n$. 6) The chromatic number of $\chi(shack(K_{(n,n)}, v_{(j,i)}, t)) = \chi(K_{n,n}) = 2$. 7) The shackle product graph produces t component and the generator graph as a subgraph of the shackle graph. The degrees of the generating graph vertex stand except for the linkage vertices. Chromatic number graph shackle equal to generator subgraph. Based on the shackle graph chromatic number, some further research can be recommended. Explore several types of chromatic numbers such as game chromatic numbers, multichromatic numbers, and star chromatic numbers.

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