Zero : Jurnal Sains, Matematika, dan Terapan
E-ISSN : 2580-5754; P-ISSN : 2580-569X
Volume 10, Number 2, 2026
DOI: 10.30829/zero.v10i2.31413
Page: 871-880



Distance−{0, 1} Vertex Irregular Total Labeling of Triangular Book Graphs

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Article Info

Article history:

Accepted: 10 October 2026

Keywords:

Graph Labeling;
Irregularity Strength;
Total Distance Labeling;
Triangular Book Graph;
Vertex Irregularity.

ABSTRACT

Irregularity strength labelings quantify, by a single integer, how far a graph is from admitting vertices with pairwise distinct weights, and total labelings sharpen this measure by letting both vertices and edges carry labels. Distance-{0,1} vertex irregular total labeling is a variant of total labeling theory in which the weight of a vertex is formed from its own label, the labels of its neighboring vertices, and the labels of its incident edges, and every vertex is required to have a distinct weight. This study investigates the distance-{0,1} vertex irregular total labeling of triangular book graphs and determines the total distance-{0,1} vertex irregularity strength, i.e. the minimum value of the maximum label k for which such a labeling exists. A constructive analytical approach is employed by formulating vertex-weight expressions, deriving a lower bound, and constructing a labeling function that attains this bound. The results establish the explicit formula $tdis_{\{0,1\}}(B_n^{(3)}) = 3$ for n = 1 and $tdis_{\{0,1\}}(B_n^{(3)}) = \lceil (n+2)/3 \rceil$ for $n \geq 2$, where $B_n^{(3)}$ denotes the triangular book graph with n pages. The graph with one page $(n = 1)$ is identified as a special case requiring a larger maximum label than the general lower bound, because with only one page vertex the three edge/vertex labels rather than by the pigeonhole argument that yields the general formula for $n \geq 2$. These findings extend the study of distance vertex irregular total labeling for book graphs and provide a foundation for future research on other graph classes and distance sets.

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1. INTRODUCTION

Graph labeling is a branch of graph theory that studies the assignment of numbers to the elements of a graph, such as vertices and edges, according to certain rules, in order to represent the mathematical structure of a graph [1], [2], [3]. As the field has developed, graph labeling has come to focus not only on the assignment of labels but also on distinguishing vertices that share a similar or symmetric structure [1], [4], [5]. One such development is distance irregular labeling, first introduced by Slamin [6], who defined the weight of a vertex as the sum of the labels of the vertices at distance one from it. This concept was subsequently extended by Bong, Lin, and Slamin [7] for cycle and wheel graphs, and further developed for various other graph classes, such as tree graphs and other graph families [8], [9].

To state these historical relationships precisely, five closely related but distinct notions are distinguished here. (a) Distance irregular labeling labels only vertices, and the weight of a vertex is the sum of the labels of the

vertices in its open neighborhood, i.e. at distance exactly one. (b) Inclusive distance vertex-irregular labeling extends (a) by also including the vertex's own label in its weight, so the weight is computed over the closed neighborhood. (c) Total vertex-irregular labeling labels both vertices and edges but, unlike (a)–(b), does not reference distance explicitly: the weight combines a vertex's own label with the labels of its incident edges. (d) Distance vertex-irregular total labeling (DVITL), introduced by Wijayanti et al. and adopted as the general framework of this study, labels both vertices and edges and defines the weight of a vertex as the sum of the label of the vertex itself, the labels of its neighboring vertices, and the labels of its incident edges (Definition 2.2). (e) Distance-{0,1} labeling denotes the specific case of DVITL in which the distance set is restricted to $D = \{0,1\}$, so that only the vertex itself (distance 0) and its immediate neighbors (distance 1) contribute to the weight (Definition 2.3); this is precisely the object investigated in the present study, with associated parameter $tdis_{\{0,1\}}(G)$.

The concept of total irregular labeling itself has been widely studied for various graph classes, such as prism graphs and arithmetic staircase graphs, before developing toward labelings that take the distance between vertices into account [10], [11]. Distance labeling subsequently developed into distance vertex irregular labeling, which requires every vertex to have a distinct weight [12], [13], [14]. However, for graphs with a symmetric structure, obtaining entirely distinct weights is not always straightforward, since several vertices may share the same distance pattern [12], [15]. To address this issue, total distance labeling was developed, in which both vertices and edges contribute to the formation of vertex weights [16], [17], [18]. One such form is the distance vertex irregular total labeling (DVITL), introduced by Wijayanti et al.[19] which defines the weight of a vertex as the sum of the label of the vertex itself, the labels of its neighboring vertices, and the labels of the edges connecting them. The total distance vertex irregularity strength (tdis) is the central parameter underlying this study. DVITL has subsequently been studied for several other graph classes, such as corona product graphs, alongside variations of other total irregular labelings such as modular total irregular labeling [16], [20].

Research on distance vertex irregular total labeling has been carried out for various graph classes, such as corona graphs, tree graphs, and graphs resulting from certain operations such as the join product [16], [8], [21]. Specifically for book graphs, several labeling variants have been studied, including irregular total edge labeling for double and triple book graphs, as well as inclusive distance labeling [22], [23], [24]. Ammelia & Agung [24] examined the inclusive vertex irregularity strength of book graphs; however, the weight definition used therein sums only vertex and edge labels without explicitly accounting for the labels of neighboring vertices. This line of research has continued to be developed, for instance by Hadiputra et al.[25] for a broader class of graphs. Meanwhile, the present study investigates distance-{0,1}vertex irregular total labeling, which incorporates edge labels into the vertex weight, thereby providing an additional degree of freedom for distinguishing weights. This difference in definition has implications for the resulting minimum value of the maximum label k for which a total k-labeling exists, with labels drawn from $\{1, 2, \ldots, k\}$ (see Definition 2.3). To date, distance-{0,1} vertex irregular total labeling for the triangular book graph $\left(B_n^{(3)}\right)$ has not, to the authors' knowledge, previously been determined, including in studies of irregular edge labeling for book graphs, which focus on edge labels without accounting for the distance between vertices.

Rather than enumerate graph classes in isolation, Table 1 below contrasts what is already known about irregularity-strength parameters for book graphs and for the general distance vertex irregular total labeling (DVITL) framework, so that the specific gap addressed by this study is stated directly rather than distributed across several sentences.

Table 1. Comparison of known irregularity-strength results for book graphs and DVITL-related studies

Source	Graph Family	Labeling/Weight Definition	Know Irregularity Strength
Ratnasari et al. [22], [23]	Book graphs (q-tuple, several book graphs)	Edge irregular total k-labeling: weight from an edge and its two endpoint labels	Total edge irregularity strength determined
Ammelia & Agung [24]	Book graphs	Inclusive distance vertex irregular labeling: weight sums only the vertex's own label and its incident edge labels (neighboring-vertex labels are not included)	Inclusive distance vertex irregularity strength determined
Wijayanti et al. [16], [19]	Corona product graphs; general DVITL framework	Distance vertex irregular total labeling (DVITL): weight sums the vertex's own label, neighboring-vertex labels, and incident-edge labels	Total distance vertex irregularity strength determined for corona products

Source	Graph Family	Labeling/Weight Definition	Know Irregularity Strength
This study	Triangular book graph $B_n^{(3)}$	Distance-{0,1} DVITL (Definition 2.3): weight sums the vertex's own label, all neighboring-vertex labels, and all incident-edge labels	$tdis_{\{0,1\}}(B_n^{(3)})$ determined exactly (Theorem 3.3)

Within this line of research, the literature specifically addressing book graphs remains comparatively narrow: existing studies on book graphs and triangular book graphs have concentrated on edge irregularity strength and inclusive distance vertex-irregularity strength, while total distance vertex-irregular labeling in the distance-{0,1} sense has not previously been examined for this class. Based on the discussion above, this study focuses on distance-{0,1}vertex irregular total labeling of the triangular book graph. Stated in explicit mathematical terms, the novelty of this study comprises four elements: (i) an explicit expression for the vertex weights of the graph $(B_n^{(3)})$; (ii) the exact value of the parameter based on the definition of distance-{0,1} total labeling, together with a sharper lower-bound mechanism obtained by exploiting the structure of page-vertex weights; (iii) an explicit labeling construction that attains this lower bound for all $n \geq 2$; and (iv) the exceptional behavior at the special case $n = 1$. The scientific contribution of this study is the derivation of an explicit formula for the minimum value of the maximum label that yields mutually distinct vertex weights on the triangular book graph. Accordingly, the objective of this study is to determine the value of the distance-{0,1} vertex irregular total labeling of the triangular book graph.

2. RESEARCH METHOD

This study employs a constructive analytical approach applied to the triangular book graph in order to determine the value of its distance-{0,1} vertex irregular total labeling. The proof strategy proceeds in three steps: first, the vertex weight expressions are derived from the definition of the labeling; second, a lower bound on the maximum label k is derived by a counting argument applied to the page vertex weight alone, with a separate bound on the spine vertex weights; third, an explicit labeling attaining this lower bound is constructed and verified branch by branch for $n \geq 2$, with the exceptional case $n = 1$ treated separately by direct case analysis.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For every vertex $v \in V(G)$, the neighborhood of v is denoted $N(v)$, the closed neighborhood of v is denoted $N[v] := N(v) \cup v$, and the degree of v is denoted $\deg(v) = |N(v)|$ [26]. If $\delta(G) \leq \deg(v) \leq \Delta(G)$ for every $v \in V(G)$,then $\delta(G)$ and $\Delta(G)$ denote, respectively, the minimum and maximum degree of the graph G [27]. For any two vertices $u, v \in V(G)$, the distance$d(u, v)$ is defined as the length of the shortest path connecting u and v [28], [29].

The definitions of degree and distance above follow standard graph-theoretic convention; the citations accompanying them are used only to source these general notions and are not relied upon for the specific labeling parameter under study. The theoretical foundation specific to this study—namely the definitions of total k-labeling, distance vertex irregular total labeling, and the distance-{0,1} vertex irregular total labeling parameter—rests on Wijayanti et al. [19], Ammelia & Agung [24], and the general distance-irregularity literature [6], [7], as stated precisely in Definitions 2.2–2.3 below.

Definition 2.1. [30] The triangular book graph with n pages, denoted $(B_n^{(3)})$, is the graph obtained from n triangles that share one common edge. Let xy denote this common edge; then

$$V(B_n^{(3)}) = \{x, y\} \cup \{v_i | 1 \leq i \leq n\} \text{ and } E(B_n^{(3)}) = \{xy\} \cup \{xv_i, yv_i : 1 \leq i \leq n\}. \quad (1)$$

The vertices x and y are called the spine vertices, while v_i is called a page vertex. An illustration of the graph for $n = 1, 2, 3$ is given in Figure 1.

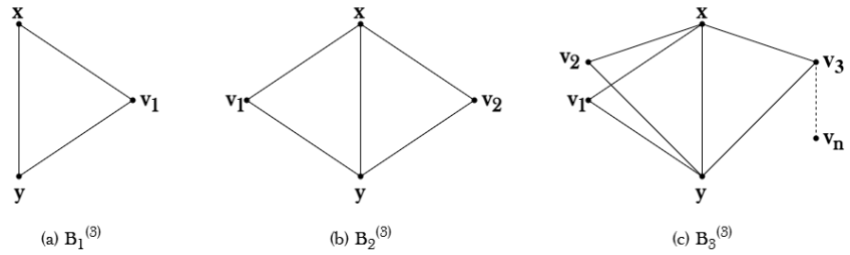


Figure 1. Triangular book graph $B_n^{(3)}$ for $n = 1, 2, 3$, illustrating the general pattern for arbitrary n; the spine vertices x, y and the page vertices vi are labeled directly on the figure.

Definition 2.2. [19] Let $G = (V(G), E(G))$ be a simple connected graph. A mapping $\sigma: V(G) \cup E(G) \rightarrow 1,2, \ldots, k$ is called a total k-labeling. For every $v \in V(G)$, the weight of vertex v under σ is defined as

$$w_\sigma(v) = \sum_{u\in N[v]} \sigma(u) + \sum_{e\in E(v)} \sigma(e) \quad (2)$$

where $N[v] = v \cup N(v)$ and $E(v)$ is the set of edges incident to v. Since $N[v] = \{v\} \cup N(v)$ is a set, v is counted in this union exactly once regardless of the size of $N(v)$; consequently, in the sum defining $w\sigma(v)$, the distance-0 contribution is exactly the single term $\sigma(v)$, and it is not added again elsewhere in the expression.

Definition 2.3. [19] A total k-labeling σ is called a distance-$\{0,1\}$ vertex irregular total labeling if $w_\sigma(u) \neq w_\sigma(v)$ for every $u, v \in V(G)$ with $u \neq v$. The value $tdis_{\{0,1\}}(G)$ is defined as the minimum value of k such that G admits a distance-$\{0,1\}$ vertex irregular total labeling with maximum label k.

Before stating the lemmas, the mathematical framework used throughout the remainder of this paper is fixed explicitly, following Wijayanti et al. [19]. The graph $B_n^{(3)}$ is simple and connected for every $n \geq 1$. A total k-labeling is a mapping $\sigma : V(G) \cup E(G) \to \{1,2,\dots,k\}$; labels are positive integers, need not be distinct across different vertices or edges, and may repeat. The maximum label k means that every label used lies in $\{1,\dots,k\}$, without requiring that k itself be attained by some label (i.e. σ need not be surjective onto $\{1,\dots,k\}$); this matches the convention in [19]. The parameter $tdis_{\{0,1\}}(G)$ (Definition 2.3) is the minimum such k for which a distance-$\{0,1\}$ vertex irregular total labeling of G exists. Under this framework, Lemma 2.4 gives closed-form expressions for the vertex weights, which are used throughout Section 3.

Lemma 2.4 For the triangular book graph $(B_n^{(3)})$ with $n \geq 1$, under a total labeling σ the vertex weights are

$$w_\sigma(x) = \sigma(x) + \sigma(y) + \sum_{i=1}^{n} \sigma(v_i) + \sigma(xy) + \sum_{i=1}^{n} \sigma(xv_i) \quad (3)$$

$$w_\sigma(y) = \sigma(y) + \sigma(x) + \sum_{i=1}^{n} \sigma(v_i) + \sigma(xy) + \sum_{i=1}^{n} \sigma(y v_i) \quad (4)$$

and for every $i = 1,2,\dots,n$,

$$w_\sigma(v_i) = \sigma(v_i) + \sigma(x) + \sigma(y) + \sigma(xv_i) + \sigma(yv_i), \quad (5)$$

Proof. By Definition 2.2, $N[x] = \{x, y, v_1, v_2, \dots, v_n\}$. Then

$$w_\sigma(x) = \sigma(x) + (\sigma(y) + \sigma(xy)) + \sum_{i=1}^{n} (\sigma(v_i) + \sigma(xv_i)),$$

which yields equation (3). Equations (4) and (5) for y and vi are obtained analogously. ■

3. RESULT AND ANALYSIS

This section determines the value of the distance-$\{0,1\}$ vertex irregular total labeling of the triangular book graph $\left(B_n^{(3)}\right)$ The discussion begins with the derivation of a lower bound, followed by the construction of a labeling that attains this bound. The main results are stated in Lemma 3.1, Lemma 3.2, and Theorem 3.3.

Lemma 3.1 For the triangular book graph $B_n^{(3)}$ with $n \geq 1$,

$$tdis_{\{0,1\}}\left(B_n^{(3)}\right) \geq \left\lceil \frac{n+2}{3} \right\rceil \quad (6)$$

Proof. Let σ be an arbitrary total k-labeling of $B_n^{(3)}$. By Lemma 2.4 for every $i = 1, 2, \dots, n$,

$$w_\sigma(v_i) = [\sigma(x) + \sigma(y)] + [\sigma(v_i) + \sigma(xv_i) + \sigma(yv_i)].$$

Since $\sigma(x) + \sigma(y)$ is the same constant for every i, the weights $w_\sigma(v_1), \dots, w_\sigma(v_n)$ are mutually distinct if and only if the values $t_i := \sigma(v_i) + \sigma(xv_i) + \sigma(yv_i)$ are mutually distinct for $i = 1, 2, \dots, n$. Since $\sigma(v_i), \sigma(xv_i), \sigma(yv_i) \in \{1, 2, \dots, k\}$, ti is a sum of three numbers drawn from this set, so $t_i \in \{3, 4, \dots, 3k\}$, a set of $3k - 2$ consecutive integers. In order for the n values $t_1, \dots, t_n$ to be chosen as mutually distinct elements of a set of size $3k - 2$, it is required that

$$3k - 2 \geq n \Leftrightarrow k \geq \frac{n+2}{3}.$$

Since k is an integer, $k \geq \lceil (n + 2)/3 \rceil$. As this holds for every total k-labeling that renders all vertex weights distinct, it follows that $tdis_{\{0,1\}}\left(B_n^{(3)}\right) \geq \lceil (n+2)/3 \rceil$. ■

Three remarks sharpen the reading of this counting argument. First, the values t_i are required to be pairwise distinct precisely because the page-vertex weights $w_\sigma(v_i)$ must be pairwise distinct under a distance-$\{0,1\}$ vertex irregular total labeling (Definition 2.3); the argument does not assume distinctness independently of this requirement. Second, the bound uses only the size of the containing set $\{3,4,\dots,3k\}$, and it is not claimed that every integer in this set is attained by some t_i; the argument is a counting (pigeonhole) bound, not an attainability statement. Third, because of this, the bound $k \geq \lceil (n+2)/3 \rceil$ need not be sharp in general; its sharpness for $n \geq 2$, together with the role played by the spine vertices, is addressed once and in full in Remark 3.1 below rather than repeated here.

Lemma 3.2 (Spine vertex weight bound). For the triangular book graph $B_n^{(3)}$ with $n \geq 1$ and any total labeling σ of $B_n^{(3)}$, the spine vertices satisfy

$$w_\sigma(x) \geq 2n + 3 \text{ and } w_\sigma(y) \geq 2n + 3 \quad (7)$$

Proof. By Lemma 2.4, $w_\sigma(x) = \sigma(x) + \sigma(y) + \sigma(xy) + \sum_{i=1}^{n} \sigma(v_i) + \sum_{i=1}^{n} \sigma(xv_i)$, a sum of $2n + 3$ labels, each at least 1 since every label is a positive integer; hence $w_\sigma(x) \geq 2n + 3$. The argument for $w_\sigma(y)$ is analogous, using the expression for $w_\sigma(y)$ in Lemma 2.4. ■

Remark 3.1 (scope of the lower bound). The inequality in Lemma 3.1 arises purely from requiring the n page-vertex weights to be pairwise distinct, and is valid for every $n \geq 1$; by itself it does not account for the possible additional constraint coming from the spine vertices x, y. For $n = 1$ this omission matters: the counting bound gives only $tdis_{\{0,1\}}(B_1^{(3)}) \geq \lceil 3/3 \rceil = 1$, yet Theorem 3.3 (Case 1) shows the true value is 3, because with a single page vertex the distinctness of $w_\sigma(x)$, $w_\sigma(y)$ and $w_\sigma(v_1)$ is governed directly by the small set of labels $\{\sigma(xy), \sigma(xv_1), \sigma(yv_1)\}$ rather than by a pigeonhole argument on n values. For $n \geq 2$, by contrast, Lemma 3.2 gives $w_\sigma(x), w_\sigma(y) \geq 2n + 3$, while the construction in the proof of Theorem 3.3 (Case 2, part (i)) attains page-vertex weights only in the range $\{5, \ldots, n + 4\}$; since $2n + 3 > n + 4$ whenever $n \geq 2$, the spine vertices cannot coincide in weight with any page vertex under that construction, so the page-count bound of Lemma 3.1 is genuinely the operative (and attained) constraint for $n \geq 2$. This is verified explicitly in parts (ii)–(iii) of the proof of Theorem 3.3 below.

As a direct check on the boundary cases $n = 1, 2, 3$, the formula is verified explicitly. For $n = 1$, Theorem 3.3 (Case 1) shows $k = 3$ is required, exceeding the counting bound $\lceil 3/3 \rceil = 1$, as explained above. For $n = 2$, $k = \lceil (2 + 2)/3 \rceil = 2$; taking $\sigma(x) = \sigma(y) = \sigma(xy) = 1, \sigma(v_1) = \sigma(v_2) = 1, \sigma(xv_1) = \sigma(xv_2) = 1, \sigma(yv_1) = 1, \sigma(yv_2) = 2$ (an instance of the general construction below) gives $w_\sigma(v_1) = 5, w_\sigma(v_2) = 6, w_\sigma(x) = 7, w_\sigma(y) = 8$, which are mutually distinct, confirming $k = 2$ suffices and is attained. For $n = 3, k = \lceil (3 + 2)/3 \rceil = 2$; taking $\sigma(v_1) = \sigma(v_2) = \sigma(v_3) = 1, \sigma(xv_1) = \sigma(xv_2) = 1, \sigma(xv_3) = 2, \sigma(yv_1) = 1, \sigma(yv_2) = 2, \sigma(yv_3) = 2$ gives $w_\sigma(v_1) = 5, w_\sigma(v_2) = 6, w_\sigma(v_3) = 7, w_\sigma(x) = 10, w_\sigma(y) = 11$, again mutually distinct. Both instances agree with the general construction of Theorem 3.3 (Case 2) and its formula $w_\sigma(v_i) = i + 4$.

Theorem 3. 3 Let $B_n^{(3)}$ be the triangular book graph. Then

$$tdis_{\{0,1\}}(B_n^{(3)}) = \begin{cases} 3, & n = 1, \\ \left\lceil \dfrac{n+2}{3} \right\rceil, & n \geq 2. \end{cases} \quad (8)$$

Proof. The graph $B_n^{(3)}$ has $n + 2$ vertices in total (the two spine vertices x, y and the n page vertices $v_1, \ldots, v_n$), so establishing the claimed value of $tdis_{\{0,1\}}$ requires verifying that all $n + 2$ of these vertex weights can be made mutually distinct at the stated maximum label, and no fewer. The proof is divided into two cases, and each case verifies distinctness of all $n + 2$ weights explicitly.

(Case 1) For $n = 1$. The graph $B_n^{(3)}$ has vertex set $\{x, y, v_1\}$ and edge set $\{xy, xv_1, yv_1\}$. From Lemma 2.4,

$$w(x) = \sigma(x) + \sigma(y) + \sigma(v_1) + \sigma(xy) + \sigma(xv_1),$$
$$w(y) = \sigma(x) + \sigma(y) + \sigma(v_1) + \sigma(xy) + \sigma(yv_1),$$
$$w(v_1) = \sigma(x) + \sigma(y) + \sigma(v_1) + \sigma(xv_1) + \sigma(yv_1).$$

Let $S = \sigma(x) + \sigma(y) + \sigma(v_1) + \sigma(xy)$, then

$w(x) = S + \sigma(xv_1)$, $w(y) = S + \sigma(yv_1)$, $w(v_1) = S + \sigma(xv_1) + \sigma(yv_1) - \sigma(xy)$,

The appearance of the term $\sigma(xy)$ here follows directly from the definitions and is made explicit as follows: by Lemma 2.4, $w(v_1) = \sigma(x) + \sigma(y) + \sigma(v_1) + \sigma(xv_1) + \sigma(yv_1)$, which contains no $\sigma(xy)$ term. Since $S = \sigma(x) + \sigma(y) + \sigma(v_1) + \sigma(xy)$ already counts $\sigma(xy)$ once, writing $w(v_1)$ in terms of S requires adding $\sigma(xv_1) + \sigma(yv_1)$ and then subtracting the $\sigma(xy)$ that S contributes but $w(v_1)$ does not actually contain: $w(v_1) = [\sigma(x) + \sigma(y) + \sigma(v_1) + \sigma(xy)] + \sigma(xv_1) + \sigma(yv_1) - \sigma(xy) = S + \sigma(xv_1) + \sigma(yv_1) - \sigma(xy)$. This cancellation is the reason $\sigma(xy)$ still influences $w(v_1)$ despite not being incident to v_1, and it is exactly what drives the collision in each branch of Table 2 below. We now show that no labeling with maximum label $k = 2$ satisfies the vertex-irregularity requirement.

Suppose $k = 2$, so every label takes the value 1 or 2. For $w(x) \neq w(y)$ it is necessary that $\sigma(xv_1) \neq \sigma(yv_1)$, since labels take only the values 1 and 2 $\{\sigma(xv_1), \sigma(yv_1)\} = \{1,2\}$. Consider the two possible values of $\sigma(xy)$:

a. If $\sigma(xy) = 1$, then $w(v_1) = S + (1 + 2) - 1 = S + 2$. Meanwhile $w(x) = S + 1$ and $w(y) = S + 2$, in this $w(y) = w(v_1) = S + 2$ a repeated weight — a contradiction.
b. If $\sigma(xy) = 2$, then $w(v_1) = S + (1 + 2) - 2 = S + 1$. Meanwhile $w(x) = S + 1$ and $w(y) = S + 2$, so $w(y) = w(v_1) = S + 1$ again a repeated weight — a contradiction

Both branches are summarized compactly in Table 2, where $S = \sigma(x) + \sigma(y) + \sigma(v_1) + \sigma(xy)$; in each branch a collision is immediate.

Table 2. Case analysis for $\sigma(xy) \in 1,2$ showing the resulting collision when n=1 and k=2

$\sigma(xy)$	$\{\sigma(xv_1)\},\sigma(yv_1)\}$	$w(x)$	$w(y)$	$w(v_1)$	Collision
1	{1,2}	$S+1$	$S+2$	$S+2$	$w(y)=w(v_1)$
2	{1,2}	$S+1$	$S+2$	$S+1$	$w(x)=w(v_1)$

Hence no labeling with $k=2$ yields all distinct weights, and thus $k \geq 3$.

Construction for $k=3$, take $\sigma(x)=\sigma(y)=\sigma(v_1)=\sigma(xy)=1, \sigma(xv_1)=2, \sigma(yv_1)=3$, then $w(x)=6$, $w(y)=7, w(v_1)=8$. all three weights distinct. Hence $\text{tdis}_{\{0,1\}}B_1^{(3)}=3$. For clarity, the six labels of this construction and the resulting weights are displayed in Table 3 and Table 4.

Table 3. Labels of the distance-{0,1} vertex irregular total labeling construction for n=1,k=3

Label	$\sigma(x)$	$\sigma(y)$	$\sigma(v_1)$	$\sigma(xy)$	$\sigma(xv_1)$	$\sigma(yv_1)$
Value	1	1	1	1	2	3

Table 4. Vertex Weight of the construction for n=1,k=3

Vertex	Weight formula (Lemma 2.4)	Computed value
x	$\sigma(x)+\sigma(y)+\sigma(v_1)+\sigma(xy)+\sigma(xv_1)$	$1+1+1+1+2=6$
y	$\sigma(x)+\sigma(y)+\sigma(v_1)+\sigma(xy)+\sigma(yv_1)$	$1+1+1+1+3=7$
v_1	$\sigma(x)+\sigma(y)+\sigma(v_1)+\sigma(xv_1)+\sigma(yv_1)$	$1+1+1+2+3=8$

(Case 2) For $n \geq 2$, by Lemma 3.1,

$$\text{tdis}_{\{0,1\}}(B_n^{(3)}) \geq \left\lceil \frac{n+2}{3} \right\rceil.$$

It remains to show that this lower bound is attained. Take $k=\lceil (n+2)/3 \rceil$, so that $n \leq 3k-2$. Define a labeling σ on $B_n^{(3)}$ with $\sigma(x)=\sigma(y)=\sigma(xy)=1$, and

$$\sigma(v_i)=\begin{cases} 1, & 1 \leq i \leq 2k-1, \\ i-2k+2, & 2k \leq i \leq n, \end{cases}$$

$$\sigma(xv_i)=\begin{cases} 1, & 1 \leq i \leq k, \\ i-k+1, & k+1 \leq i \leq 2k-1, \\ k, & 2k \leq i \leq n, \end{cases}$$

and

$$\sigma(yv_i)=\begin{cases} i, & 1 \leq i \leq k, \\ k, & k+1 \leq i \leq n. \end{cases}$$

Since $n \leq 3k-2$, all labels above lie within $\{1,2,\ldots,k\}$.

We verify this explicitly at each branch boundary. For $\sigma(v_i)$: on the first branch the value is fixed at 1, and on the branch $2k \leq i \leq n$ the value increases from 2 at $i=2k$ up to $n-2k+2$ at $i=n$; since $n \leq 3k-2$ this maximum satisfies $n-2k+2 \leq k$, so $\sigma(v_i) \in \{1,\ldots,k\}$ throughout. For $\sigma(xv_i)$: the first branch is fixed at 1; the middle branch $k+1 \leq i \leq 2k-1$ increases monotonically from 2 at $i=k+1$ to k at $i=2k-1$; and the last branch is fixed at k; hence $\sigma(xv_i) \in \{1,\ldots,k\}$ throughout. For $\sigma(yv_i)$: the first branch equals i, ranging from 1 to k as i runs over $1 \leq i \leq k$, and the second branch is fixed at k; hence $\sigma(yv_i) \in \{1,\ldots,k\}$ throughout. Thus every label assigned by σ lies in $\{1,2,\ldots,k\}$, so σ is indeed a valid total k-labeling. We now compute the resulting vertex weights.

(i) Weight of the page vertices. By Lemma 2.4, $w_\sigma(v_i)=\sigma(v_i)+\sigma(xv_i)+\sigma(yv_i)+2$ for every i, checking the three branches of i:

a. $1 \leq i \leq k : w_\sigma(v_i)=1+1+i+2=i+4.$

b. $k+1 \leq i \leq 2k-1 : w_\sigma(v_i)=1+(i-k+1)+k+2=i+4.$

c. $2k \leq i \leq n: w_\sigma(v_i)=(i-2k+2)+k+k+2=i+4.$

Hence $w_\sigma(v_i)=i+4$ for every $i=1,\ldots,n$, so $\{w_\sigma(v_i)|1 \leq i \leq n\}=\{5,6,\ldots,n+4\}$,all mutually distinct. The formula $w_\sigma(v_i)=i+4$ is further verified explicitly at every branch transition and at the extremal values of k and n. At $i=k$ (branch 1): $w=1+1+k+2=k+4$. At $i=k+1$ (branch 2): $w=1+(k+1-k+1)+k+2=k+5$, consistent with $i+4$ on both sides of the transition. At $i=2k-1$ (branch 2): $w=1+(2k-1-k+1)+k+2=2k+3=(2k-1)+4$. At $i=2k$ (branch 3): $w=(2k-2k+2)+k+k+2=2k+4=(2k)+4$, again consistent. When $k=2$, branch 2 reduces to the single index $i=3=k+1=2k-1$, and $w=1+(3-2+1)+2+2=7=3+4$, as required. When $n=2k-1$, branch 3$(2k \leq i \leq n)$ is vacuous and the construction uses only branches 1–2, which remain valid. When $n=3k-2$ (the largest n admitted by $k=\lceil (n+2)/3 \rceil$), branch 3 reaches $i=n=3k-2$, giving $w=(3k-2-2k+2)+2k+2=3k+2=(3k-2)+4$, confirming the formula holds up to the extremal case.

(ii) Weight of vertices x and y. Since every term $\sigma(v_i) \geq 1$ and $\sigma(xv_i) \geq 1$ (n term each), $w_\sigma(x) = \sigma(x) + \sigma(y) + \sum_{i=1}^{n} \sigma(v_i) + \sigma(xy) + \sum_{i=1}^{n} \sigma(xv_i) \geq 1 + 1 + n + 1 + n = 2n + 3$.
For $n \geq 2, 2n + 3 > n + 4$,so $w_\sigma(x)$ differs from every $w_\sigma(v_i)$ (whose maximum value is n+4).

(iii) By Lemma 2.4,

$$w_\sigma(y) - w_\sigma(x) = \sum_{i=1}^{n} \big(\sigma(yv_i) - \sigma(xv_i)\big).$$

Examining each term, let $d_i := \sigma(yv_i) - \sigma(xv_i)$:

a. $1 \leq i \leq k : d_i = i - 1 \geq 0$ and $(d_1 = 0, d_2 = 1)$.

b. $k + 1 \leq i \leq 2k - 1 : d_i = k - (i - k + 1) = 2k - i - 1 \geq 0$ since $i \leq 2k - 1$.

c. $2k \leq i \leq n : d_i = k - k = 0$.

Hence $d_i \geq 0$ for every i. Since $n \geq 2$, $k = \lceil (n + 2)/3 \rceil \geq 2$,so the indexs $i = 2$ lies in the first branch and gives $d_2 = 1 > 0$. Since all other terms are ≥ 0,

$$w_\sigma(y) - w_\sigma(x) = \sum_{i=1}^{n} d_i \geq d_2 = 1 > 0,$$

so $w_\sigma(y) > w_\sigma(x)$, and with (ii), $w_\sigma(y)$ also differs from every $w_\sigma(v_i)$.

The lower bound $n + 4$ used above to rule out a collision between $w_\sigma(x), w_\sigma(y)$ and the page-vertex weights is applied specifically to the labeling σ constructed in this proof, not to an arbitrary total k-labeling; all labels of σ are positive integers by construction, which is what makes the inequality $w_\sigma(x) \geq 2n > 3\,n + 4$ strict for $n \geq 2$.

The difference $w_\sigma(y) - w_\sigma(x)$ can in fact be computed exactly, rather than merely bounded below by $d_2 = 1$. Summing di over the first branch $(1 \leq i \leq k), \Sigma(i - 1) = k(k - 1)/2$. Summing over the second branch $(k + 1 \leq i \leq 2k - 1)$, the values $2k - i - 1$ run over $0, 1, \ldots, k - 2$, so the sum is $(k - 1)(k - 2)/2$. The third branch $(2k \leq i \leq n)$ contributes 0. Adding these,

$$w_\sigma(y) - w_\sigma(x) = \frac{k(k-1)}{2} + \frac{(k-1)(k-2)}{2} = (k-1)^2$$

Since $n \geq 2$ forces $k = \lceil (n + 2)/3 \rceil \geq 2$, this gives $w_\sigma(y) - w_\sigma(x) = (k - 1)^2 \geq 1$, confirming strict positivity by an exact closed form rather than a term-by-term bound. (Check: $k = 2$ gives $(k - 1)^2 = 1$, matching $d_2 = 1$ computed directly above; $k = 3$ gives $(k - 1)^2 = 4$.)

From (i)-(iii), all $n + 2$ vertex weight of $B_n^{(3)}$ are mutually distinct under the maximum label $k = \lceil (n + 2)/3 \rceil$, hence

$$\text{tdis}_{\{0,1\}} B_n^{(3)} \leq k = \left\lceil \tfrac{n+2}{3} \right\rceil.$$

Combined with the lower bound of Lemma 3.1, for $n \geq 2$ it follows that

$$\text{tdis}_{\{0,1\}}\big(B_n^{(3)}\big) = \left\lceil \tfrac{n+2}{3} \right\rceil. \blacksquare$$

4. CONCLUSION

This study has examined the distance-{0,1} vertex irregular total labeling of the triangular book graph $B_n^{(3)}$. Based on the analysis of the vertex weight expressions and the labeling construction provided, it is obtained that

$$\text{tdis}_{0,1}\big(B_n^{(3)}\big) = \begin{cases} 3, & n = 1, \\ \left\lceil \dfrac{n+2}{3} \right\rceil, & n \geq 2, \end{cases}$$

This result shows that for $n = 1$, the value of tdis is 3 (greater than the general lower bound, due to the very small characteristic structure of the graph), whereas for $n \geq 2$ the exact value of $tdis_{\{0,1\}}$ is given by the ceiling formula above, which in particular shows that $tdis_{\{0,1\}}$ grows linearly with the number of pages, informally at a rate of approximately $\frac{n}{3}$, since $\lceil (n + 2)/3 \rceil$ is asymptotically $n/3$. This pattern reflects the fact that each page vertex contributes three labels to the formation of its weight, so that the n page-vertex weights are constrained to lie within a range of $3k - 2$ consecutive integers for maximum label k (Lemma 3.1); this containing-set size yields the counting lower bound, and the constructive proof of Theorem 3.3 (Case 2) confirms that this bound is exactly attained, with the page-vertex weights realizing the sub-range $\{5, \ldots, n + 4\}$. The conclusion above reports this proven result rather than a merely conjectured one.

It is useful to relate this value to related irregularity parameters previously determined for book graphs. Ratnasari et al. [22], [23] determined the total edge irregularity strength of several book-graph families under a weight definition built only from an edge and its two endpoint labels, and Ammelia and Agung [24] determined the inclusive distance vertex irregularity strength of book graphs under a weight definition that sums a vertex's own label with its incident edge labels but not the labels of neighboring vertices. The distance-$\{0,1\}$ definition used here additionally sums the labels of all neighboring vertices (Definition 2.3), so each page-vertex weight draws on more independent labels than in either [22], [23], [24]; this additional freedom is what permits the piecewise construction of Theorem 3.3 to spread the n page-vertex weights across a range of only $3k - 2$

consecutive integers. A term-by-term numerical comparison against the exact formulas of [22], [23], [24] would require restating those results under an identical labeling convention and is left for future work, but the qualitative direction is clear: strictly more label information per vertex weight generally permits a smaller maximum label k for a fixed number of pages n. On the tightness and generality of the method: the constructive template used in the proof of Theorem 3.3, namely fixing the spine-vertex labels and distributing the page-vertex weights across piecewise-linear branches of the edge labels, does not use any feature specific to triangles beyond the existence of exactly two edges per page vertex; it is plausible that an analogous construction extends to other graphs formed by n copies of a small fixed graph sharing a common vertex or edge, such as fan graphs, friendship (windmill) graphs, or general q-tuple book graphs, provided a vertex-weight decomposition analogous to Lemma 2.4 holds. Verifying this extension for each such family, however, remains open.

Future research may be directed toward variations of the distance set, for example distance-$\{0, 1, 2\}$ or distance-$\{1, 2\}$, and toward graphs obtained from other operations, such as the corona product or amalgamation of book graphs. Accordingly, this study contributes to the development of the theory of distance vertex irregular total labeling, particularly for the class of book graphs.

ACKNOWLEDGEMENT

This section is optional and should be included only if necessary. The Acknowledgement section is used to express appreciation to individuals, institutions, or funding agencies that contributed to the completion of the research but are not listed as authors. Authors should specify funding information, research grants, or institutional support if applicable. Avoid personal or overly informal expressions of gratitude. If there was no external funding or special assistance, this section may be omitted.

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