Zero : Jurnal Sains, Matematika, dan Terapan
E-ISSN : 2580-5754; P-ISSN : 2580-569X
Volume 10, Number 2, 2026
DOI: 10.30829/zero.v10i2.29239
Page: 659-670

A Mathematical Model for Conservation Strategies of Rare Plants in North Sumatra

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Article Info

Article history:

Accepted: 10 July 2026

Keywords:

Conservation strategies;
Mathematical modeling;
Population dynamics;
Rare plants;
Stability analysis.

ABSTRACT

This study develops a nonlinear mathematical model for the conservation of rare plant populations in North Sumatra by incorporating interactions among rare plants, human populations, population pressure, and fire- and toxicant-control activities. The model is formulated as a system of four nonlinear ordinary differential equations and analyzed using positivity, boundedness, equilibrium, and stability theory. The analysis establishes four equilibrium points, including a biologically relevant coexistence equilibrium. Conditions for the existence of this equilibrium are derived, while its local and global asymptotic stability are established using the Routh–Hurwitz criterion and Lyapunov stability theory, respectively. Numerical simulations using illustrative parameter values are performed to support the analytical results and demonstrate the qualitative behavior of the system. The results provide analytical conditions under which rare plant populations can persist despite human pressure and environmental disturbances. This study is theoretical in nature and is not calibrated using ecological field data; therefore, the numerical results are intended to illustrate model dynamics rather than provide site-specific conservation predictions. The proposed framework contributes to the mathematical understanding of conservation dynamics and may serve as a foundation for future data-driven conservation studies.

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1. INTRODUCTION

Rare plants play an important role in maintaining ecosystem balance, biodiversity, and environmental stability, especially in tropical regions such as North Sumatra, which is recognized as one of Indonesia's biodiversity hotspots. The existence of rare plants not only supports ecological functions, such as providing habitat and regulating nutrient cycles, but also has economic and cultural value for local communities and contributes to human well-being through the ecosystem services they provide [1], [2].

However, rare plant populations in North Sumatra face serious threats from deforestation, agricultural expansion, human population growth, climate change, and environmental pollution [3]. The conversion of natural forests into agricultural land has been recognized as one of the major drivers of biodiversity loss in tropical ecosystems [4]. Furthermore, habitat destruction frequently leads to habitat fragmentation, resulting in smaller

and more isolated habitat patches that reduce biodiversity and alter ecosystem functioning over the long term [5] [6], [7]. Recent studies indicate that human activities, including land conversion and unsustainable exploitation of natural resources, significantly accelerate population decline and habitat fragmentation, thereby increasing the risk of extinction for rare plant species [1], [8].

In this study, rare plants refer to species in North Sumatra with limited populations, restricted distributions, or specialized habitat requirements, making them highly vulnerable to environmental disturbances and human pressures. The model represents the broader class of rare plant populations whose persistence depends on habitat quality and conservation efforts, regardless of their formal conservation status. Rare plant conservation has been studied using various mathematical models. Logistic-growth models describe population dynamics under a fixed carrying capacity, resource-depletion models incorporate human population and habitat pressure, toxicant models account for pollutant effects on species persistence, and optimal-control models evaluate conservation strategies [7], [9], [10].

More recently, Nababan et al. [11] proposed a society-inclusive dynamic conservation model for rare plants that explicitly incorporates interactions between rare plant populations, human population growth, and conservation activities. Their work demonstrated the importance of social participation in supporting plant conservation and provided valuable insights into the role of conservation strategies in maintaining population persistence. However, the model does not explicitly account for multiple environmental stressors such as man-made fire and environmental toxicants, nor does it investigate how these stressors jointly interact with conservation effort and human-induced habitat degradation to influence long-term system stability [12], [13].

This study extends previous conservation models by incorporating fire disturbance, toxicant exposure, human population pressure, and conservation effort into a unified nonlinear dynamical system. Rather than replacing existing frameworks, the model provides a broader representation of the interacting factors affecting rare plant persistence in regions such as North Sumatra. The study also distinguishes between components adapted from earlier models and the new analytical developments introduced here.

The main contribution of this work is the analytical characterization of a conservation-driven persistence threshold for rare plant populations under multiple interacting stressors. The model shows how conservation effort, human population pressure, habitat degradation, fire disturbance, and toxicant effects jointly determine population persistence. The resulting threshold provides explicit stability criteria for assessing whether conservation interventions can offset anthropogenic pressures, offering a mathematical framework that supports conservation planning and decision-making.

2. RESEARH METHOD

2.1 Mathematical Formulation

This section presents a nonlinear ordinary differential equation (ODE) model describing the interactions among rare plant population, human population, population pressure, and the control of manmade fire and toxicant activity in North Sumatra. The model is based on the following assumptions:

Assumption A1 (Logistic growth of rare plants)
In the absence of human disturbance and population pressure, the rare plant population grows according to a logistic law with intrinsic growth rate $s$and carrying capacity L. This assumption reflects the limitation of environmental resources and habitat availability, causing population growth to slow as the population approaches it carrying capacity.

Assumption A2 (Human impact on rare plants)
The interaction between rare plants and the human population follows a mass-action term BN, where rare plant depletion due to human activities is modeled by $\beta_1 BN$.

Assumption A3 (Population pressure generation) Population pressure is assumed to increase proportionally with the human population size. This assumption reflects the fact that larger human populations generally generate greater demands for land, infrastructure, and natural resources, thereby intensifying environmental pressure. Population pressure decreases naturally at rate λ_0.

Assumption A4 (Conservation response mechanism)
Fire- and toxicant-control activity increases as the rare plant population declines below its ecological potential. The ecological deficit, measured by $1-\frac{B}{L}$, reflects the population's distance from the carrying capacity L. Thus, conservation effort intensifies as Bdecreases and is modeled by the term

$$\Delta\left(1-\frac{B}{L}\right)$$

The term $\Delta\left(1-\frac{B}{L}\right)$ follows a feedback-based conservation mechanism in which intervention increases as the rare plant population declines below its ecological level, consistent with bioeconomic and ecological management models[14].

Assumption A5 (Linear conservation effect)
The interaction term $\sigma_0 BF$ represents the positive effect of fire- and toxicant-control activities, increasing with rare plant abundance and conservation effort.

Assumption A6 (Effective human population)
The term βBN represents the attraction and retention of human activities associated with rare plant resources, rather than biological reproduction.

2.2 Equation of Rare Plant Population

The rare plant population (B) is assumed to follow logistic growth, with rapid growth at low population levels and slower growth near the carrying capacity L. This logistic model has been shown to be relevant for plant populations in limited habitats [3], [4], [15]. The population is affected by natural mortality (s_0), human activities (β_1), population pressure (λ_1), and conservation support (σ_0), as described in Eq. (1):

$$\frac{dB}{dt} = s\left(1-\frac{B}{L}\right)\mathrm{B} - s_0 B - \beta_1 NB - \lambda_1 PB + \sigma_0 FB \tag{1}$$

2.3 Equation of Human Population N(t)

Human population density (N) reduces rare plant populations through direct consumption, land conversion, and habitat destruction. This effect is modeled as a proportional decrease in the rate of interaction between N and B [3], [4]. The dynamics of the human population are described by Eq. (2):

$$\frac{dN}{dt} = R\left(1-\frac{N}{K}\right)\mathrm{N} - r_0 N + \beta NB \tag{2}$$

2.4 Equation of Population Pressure P(t)

Population pressure (P) is a combination of human population growth and activities that accelerate habitat degradation. The growth rate of population pressure is denoted by λ. The natural depletion rate of population pressure is parameterized by λ_0 [3]. These dynamics are given by Eq. (3):

$$\frac{dP}{dt} = \lambda N - \lambda_0 P \tag{3}$$

2.5 Equation of Controlling Man-Made Fire and Toxicant Activity F(t)

Controlling man-made fire and toxicant activity (F) represents conservation efforts to mitigate anthropogenic disturbances. Conservation effort increases as the rare plant population declines below its ecological potential. The term $1-\frac{B}{L}$ represents the ecological deficit relative to the carrying capacity L: as Bapproaches L, intervention is limited, whereas lower Bintensifies conservation efforts. Thus, the control effort is proportional to $1-\frac{B}{L}$, as expressed in Eq. (4).

$$\frac{dF}{dt} = \Delta\left(1-\frac{B}{L}\right) - \sigma F + \delta F \tag{4}$$

Hence, by combining the above equations, the system of nonlinear differential equations governing the dynamics of the system can be written as the coupled system in Eq. (5) - (8):

$$\frac{dB}{dt} = s\left(1-\frac{B}{L}\right)\mathrm{B} - s_0 B - \beta_1 NB - \lambda_1 PB + \sigma_0 FB \tag{5}$$

$$\frac{dN}{dt} = R\left(1-\frac{N}{K}\right)\mathrm{N} - r_0 N + \beta NB \tag{6}$$

$$\frac{dP}{dt} = \lambda N - \lambda_0 P \tag{7}$$

$$\frac{dF}{dt} = \Delta\left(1-\frac{B}{L}\right) - \sigma F + \delta F \tag{8}$$

Where:

Variable

B = Rare plant population
N = Human population
P = Population pressure
F = Controlling man-made fire and toxicant activity

Parameter

s, L = Intrinsic growth rate and carrying capacity of rare plants
R, K = The growth rate and carrying capacity of human population
r_0 = Natural depletion rate of human population
β_1 = Depletion rate of rare plant population
s_0 = Natural depletion (mortality) rate of rare plants
λ_1 = Depletion rate of rare plants due to population pressure
β = Growth rate of human population due to interaction with rare plants.
λ = The growth rate of population pressure
λ_0 = Natural depletion rate of population pressure
σ = Depletion rate of fire- and toxicant-control activity
σ_0 = Conservation rate of rare plants due to fire and toxicant-control activity
Δ = Rate of efforts to conserve rare plants
δ = Conservation rate of controlling man-made fire and toxicant activities

The term $+\beta NB$represents the attraction and retention of human activity by rare plant resources, rather than biological reproduction. Therefore, Ndenotes the effective human population associated with the ecosystem instead of demographic growth. This model adopts an approach that has been validated in the literature on conservation and ecological modeling [3], [8]. This theoretical model enables quantitative analysis of ecological, social, and economic interactions affecting rare plant dynamics. Numerical simulations use representative, biologically plausible parameter values to illustrate the analytical results rather than provide calibrated predictions for North Sumatra. Calibration with field data is left for future research.

3 RESEARCH AND ANALYSIS

This section investigates the qualitative behavior of system (5)–(8) by establishing solution positivity and boundedness, determining the biologically meaningful equilibria and their existence conditions, and analyzing their local and global asymptotic stability under the following assumption.

Assumption 1. All model parameters are positive, all initial states are non-negative, and the control activity is net-dissipative, i.e.

$$\sigma > \delta.$$

The condition $\sigma > \delta$ is required because the control variable F enters Eq. (8) linearly through the net term $(\delta - \sigma)F$; if $\sigma \leq \delta$ this term is non-dissipative and F grows without bound, so no bounded equilibrium of the system can exist. Throughout, we write the system compactly as

$$\frac{dB}{dt} = f_1, \quad \frac{dN}{dt} = f_2, \quad \frac{dP}{dt} = f_3, \quad \frac{dF}{dt} = f_4 \tag{9}$$

3.1 Positivity and Boundedness of Solutions

For the model to be biologically meaningful, every solution starting in the non-negative orthant must remain non-negative and bounded.

Theorem 1 (Positivity). Under Assumption 1, the region $\mathbb{R}^4_+ = \{(B, N, P, F): B \geq 0,\ N \geq 0,\ P \geq 0,\ F \geq 0\}$ is positively invariant.

Proof. Equations (5) and (6) can be written as $\dot{B} = B\, g_1(B, N, P, F)$ and $\dot{N} = N\, g_2(B, N)$, so that

$$B(t) = B(0)\exp\left(\int_0^t g_1\, d\tau\right) \geq 0, \qquad N(t) = N(0)\exp\left(\int_0^t g_2\, d\tau\right) \geq 0, \tag{10}$$

which preserves the sign of the initial data. On the face $P = 0$, Eq. (7) gives $\dot{P} = \lambda N \geq 0$, and on the face $F = 0$, Eq. (8) gives $\dot{F} = \Delta\left(1 - \frac{B}{L}\right)$, which is non-negative whenever $B \leq L$. Hence no trajectory can leave $\mathbb{R}^4_+$ through these faces, and the orthant is positively invariant.

Theorem 2 (Boundedness). Under Assumption 1, all solutions of (2.5)–(2.8) are ultimately bounded and enter the compact region

$$\Omega = \{(B, N, P, F): 0 \le F \le F_m,\ 0 \le B \le B_m,\ 0 \le N \le N_m,\ 0 \le P \le P_m\}, \qquad (11)$$

where

$$F_m = \frac{\Delta}{\sigma-\delta}, \qquad B_m = \frac{L\,(s-s_0+\sigma_0 F_m)}{s}, \qquad N_m = \frac{K\,(R-r_0+\beta B_m)}{R}, \qquad P_m = \frac{\lambda N_m}{\lambda_0}. \qquad (12)$$

Proof. Since $0 \le 1-\frac{B}{L} \le 1$ for $0 \le B \le L$, Eq. (8) yields $\dot{F} \le \Delta - (\sigma-\delta)F$, and a standard comparison argument gives $\limsup_{t\to\infty} F(t) \le F_m$. Using $F \le F_m$ in Eq. (5) gives $\dot{B} \le B(s - s_0 + \sigma_0 F_m - s/L\,B)$, whence $\limsup_{t\to\infty} B(t) \le B_m$. Substituting $B \le B_m$ into Eq. (6) gives $\dot{N} \le N(R - r_0 + \beta B_m - R/K\,N)$, so $\limsup_{t\to\infty} N(t) \le N_m$; and from Eq. (2.7), $\dot{P} \le \lambda N_m - \lambda_0 P$, so $\limsup_{t\to\infty} P(t) \le P_m$. The coupled bounds admit finite positive solutions under Assumption 1, ensuring that all state variables remain uniformly bounded, so Ω is an attracting, positively invariant region. The subsequent analysis is therefore restricted to the biologically relevant region Ω. Substituting (12) into the expression for $M_B$yields a linear algebraic equation in M_B, which admits a finite positive solution whenever $\sigma > \delta$and $s > 0$.

3.2. Equilibrium Points

Equilibria are obtained by setting $f_1 = f_2 = f_3 = f_4 = 0$. Equations (7) and (8) provide two slaving relations,

$$P = \frac{\lambda}{\lambda_0} N, \qquad F = \frac{\Delta\left(1-\frac{B}{L}\right)}{\sigma-\delta}, \qquad (13)$$

while (6) gives $N = 0$ or $N = \frac{K}{R}(R - r_0 + \beta B)$, and (5) gives $B = 0$ or the bracket condition $s(1 - B/L) - s_0 - \beta_1 N - \lambda_1 P + \sigma_0 F = 0$. Combining these cases yields four equilibria.

(i) Total-extinction equilibrium.

$$E_0 = (0,\ 0,\ 0,\ \Delta/\sigma - \delta), \qquad (14)$$

which always exists; note $F \neq 0$ because the constant control input Δ sustains the activity even in the absence of plants.

(ii) Plant-extinction equilibrium.

$$E_1 = (0,\ K(R - r_0)/R\,,\ \lambda/\lambda_0\,K(R - r_0)/R\,,\ \Delta/\sigma - \delta), \qquad (15)$$

which exists provided $R > r_0$. This is the ecologically undesirable state in which the human population persists while the rare plant is lost.

(iii) Human-extinction equilibrium.

$E_2 = (B_2, 0, 0, F_2)$ with $F_2 = \Delta\left(1 - \frac{B_2}{L}\right)/\sigma - \delta$ and B_2 the unique root of $(s + \sigma_0\,\Delta/\sigma - \delta)(1 - B_2/L) = s_0$, which lies in $(0, L)$ whenever $0 < s_0 < s + \sigma_0\,\Delta/\sigma - \delta$.

(iv) Coexistence equilibrium

$E^* = (B^*, N^*, P^*, F^*)$. Substituting the slaving relations (3.1) and $N = N_0 + n_1 B$ into the bracket condition makes it *linear in* B, giving a unique

$$B^* = \frac{s - s_0 - A\,N_0 + \sigma_0 F_0}{\frac{s}{L} + A\,n_1 + \sigma_0 f_1}, \qquad (16)$$

with the shorthand

$$A = \beta_1 + \frac{\lambda_1 \lambda}{\lambda_0}, \quad N_0 = \frac{K(R - r_0)}{R}, \quad n_1 = \frac{K\beta}{R}, \quad F_0 = \frac{\Delta}{\sigma-\delta}, \quad f_1 = \frac{\Delta}{L(\sigma-\delta)}, \qquad (17)$$

and then $N^* = N_0 + n_1 B^*$, $P^* = \lambda/\lambda_0\,N^*$, $F^* = F_0 - f_1 B^*$. The denominator of (3.2) is always positive, so E^* exists in the interior of Ω if and only if

$$\underbrace{s + \sigma_0 F_0 s + \sigma_0 F_0}_{\text{effective growth}} \quad > \quad \underbrace{s_0 + A\,N_0}_{\text{mortality + human pressure}} \qquad \text{and} \quad B^* < L.$$

Condition (3.3) defines the persistence threshold: rare plants persist when intrinsic growth plus conservation benefit ($\sigma_0 F_0$) exceeds natural mortality and depletion (AN_0). The condition $B^* < L$ensures $F^* > 0$.

3.3. Stability of Boundary Equilibria

Stability of E_0

Evaluating the Jacobian matrix at E_0 yields the eigenvalues

$$\mu_1 = s - s_0 + \frac{\sigma_0 \Delta}{\sigma - \delta}, \mu_2 = R - r_0, \mu_3 = -\lambda_0, \mu_4 = -(\sigma - \delta).$$

Hence, E_0 is locally asymptotically stable if

$$s - s_0 + \frac{\sigma_0 \Delta}{\sigma - \delta} < 0$$

and

$$R - r_0 < 0.$$

$$\mu 1 = s - s_0 + \frac{\sigma_0 \Delta}{\sigma - \delta}, \mu 2 = R - r_0, \mu 3 = -\lambda_0, \mu 4 = -(\sigma - \delta).$$

Stability of E_1.

The equilibrium E_1 represents the extinction of the rare plant population while the human population persists. Evaluating the Jacobian matrix at E_1 yields the eigenvalues

$$\mu 1 = s - s_0 - \beta_1 N_1^* - \lambda_1 P_1^* + \sigma_0 F_1^*, \ \mu_2 = -(R - r_0), \ \mu_3 = -\lambda_0, \ \mu_4 = -(\sigma - \delta).$$

Since $\mu_2, \mu_3,$ and μ_4 are negative under the existence conditions of E_1 and Assumption 1, the local stability of E_1 is determined by μ_1. Therefore, E_1 is locally asymptotically stable if

$$s + \sigma_0 F_1^* < s_0 + \beta_1 N_1 + \lambda_1 P_1^*.$$

his condition characterizes rare plant extinction and the failure of the conservation strategy.

Stability of E_2

The invasion eigenvalue associated with the human population is

$$\mu_N = R - r_0 + \beta B_2^*$$

Therefore, a necessary condition for the local stability of E_2 is

$$R - r_0 + \beta B_2^* < 0$$

3.4. Local Stability Analysis

The Jacobian of system (5) - (8) evaluated at the coexistence equilibrium E^*, after using the equilibrium bracket conditions to simplify the diagonal entries, is

$$J(E^*) = \begin{pmatrix} -\frac{s}{L}B^* & -\beta_1 B^* & -\lambda_1 B^* & \sigma_0 B^* \\ \beta N^* & -\frac{R}{K}N^* & 0 & 0 \\ 0 & \lambda & -\lambda_0 & 0 \\ -\frac{\Delta}{L} & 0 & 0 & -(\sigma - \delta) \end{pmatrix}. \quad (18)$$

The characteristic polynomial associated with $J(E^*)$ is

$$\mu^4 + a_1\mu^3 + a_2\mu^2 + a_3\mu + a_4 = 0. \quad (19)$$

Its coefficients are given by

$$a_1 = -tr\big(J(E*)\big) = \frac{s}{L}B* + \frac{R}{K}N* + \lambda 0 + (\sigma - \delta) \quad (20)$$

$$a_2 = \frac{sR}{LK}B^*N^* + \frac{s}{L}B^*\lambda_0 + \frac{s}{L}B^*(\sigma - \delta) + \beta_1\beta B^*N^* + \frac{\Delta\sigma_0 B^*}{L} + \frac{R}{K}N^*\lambda_0 + \frac{R}{K}N^*(\sigma - \delta) + \lambda_0(\sigma - \delta), \quad (21)$$

$$a_3 = \left(\frac{sR}{LK}\right)B^*N^*\lambda_0 + \left(\frac{sR}{LK}\right)B^*N^*(\sigma - \delta) + \frac{s}{L}B^*\lambda_0(\sigma - \delta) + \beta_1\beta B^*N^*\lambda_0 +$$

$$\beta_1\beta B^*N^*(\sigma - \delta) + \left(\frac{\Delta\sigma_0 B^*}{LK}\right)B^*N^* + \left(\frac{\Delta\sigma_0 B^*}{L}\right)\lambda_0 + \lambda_1\beta\lambda B^*N^* + \frac{R}{K}N^*\lambda_0(\sigma - \delta) \quad (22)$$

and

$$a_4 = \left(\frac{\Delta\sigma_0 B^*}{LK}\right) B^* N^* \lambda_0 + (\sigma - \delta)\left[\frac{sR}{LK} B^* N^* \lambda_0 + \beta_1 \beta B^* N^* \lambda_0 + \lambda_1 \beta \lambda B^* N^*\right] \quad (23)$$

Since all parameters are positive, $B^* > 0, N^* > 0$, and Assumption 1 implies $\sigma - \delta > 0$, every term appearing in Eq. (20)–(23) is positive. Consequently,

$$a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0$$

Therefore, E^* is locally asymptotically stable provided that the Routh–Hurwitz condition (24) holds.

$$a_1 a_2 a_3 > a_3^2 + a_1^2 a_4. \quad (24)$$

Theorem 3 (Local stability). The coexistence equilibrium $E^*$is locally asymptotically stable if

$$a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, \quad a_1 a_2 a_3 > a_3^2 + a_1^2 a_4.$$

Proof. The characteristic polynomial associated with the Jacobian matrix $J(E^*)$is given by Eq. (19). According to the Routh–Hurwitz criterion for fourth-order polynomials, local asymptotic stability is ensured whenever the coefficients satisfy

$$a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0,$$

together with

$$a_1 a_2 a_3 > a_3^2 + a_1^2 a_4.$$

From Eqs. (20) - (23), each coefficient is expressed as a sum of positive quantities involving positive parameters, positive equilibrium values $B^* > 0$and $N^* > 0$, and the positive term $(\sigma-\delta)$guaranteed by Assumption 1. Therefore,

$$a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0.$$

Hence, the coexistence equilibrium $E^*$is locally asymptotically stable whenever

$$a_1 a_2 a_3 > a_3^2 + a_1^2 a_4.$$

3.5 Global Stability Analysis

To establish global asymptotic stability of E^*, we construct a Lyapunov function of mixed Goh (logarithmic) and quadratic type,

$$V = c_1\left(B - B^* - B^* \ln\frac{B}{B^*}\right) + c_2\left(N - N^* - N^* \ln\frac{N}{N^*}\right) + \frac{c_3}{2}(P - P^*)^2 + \frac{c_4}{2}(F - F^*)^2 \quad (25)$$

with positive constants c_i, $i = 1,2,3,4$.
Each logarithmic term is nonnegative and vanishes only at the coexistence equilibrium E^*. Therefore,

$$V > 0, V(E^*) = 0,$$

and V is positive definite in a neighborhood of E^*.
Introducing the translated variables

$$x = B - B^*, y = N - N^*, z = P - P^*, w = F - F^*,$$

and differentiating Valong trajectories of system (5) - (8), after substitution of the equilibrium relations, yields

$$\dot{V} = -Ax^2 - By^2 - Cz^2 - Dw^2 + (c_1\beta_1 - c_2\beta)xy + (c_1\sigma_0 L - c_4\Delta)xw - \lambda_1 xz + c_3\lambda yz, \quad (26)$$

where

$$A > 0, B > 0, C > 0, D > 0.$$

Choosing

$$c_1 = 1, c_2 = \frac{\beta_1}{\beta}, c_4 = \frac{\sigma_0 L}{\Delta}, \quad (27)$$

eliminates the mixed terms since

$$c_1\beta_1 - c_2\beta = \beta_1 - \frac{\beta_1}{\beta}\beta = 0,$$

and

$$c_1\sigma_0 L - c_4\Delta = \sigma_0 L - \frac{\sigma_0 L}{\Delta}\Delta = 0.$$

Hence,

$$\dot{V} = -Ax^2 - By^2 - Cz^2 - Dw^2 - \lambda_1 xz + c_3\lambda yz. \tag{28}$$

The quadratic form in $(x, y, z)$is associated with the symmetric matrix

$$M = \begin{pmatrix} A & 0 & \frac{\lambda_1}{2} \\ 0 & B & -\frac{c_3\lambda}{2} \\ \frac{\lambda_1}{2} & -\frac{c_3\lambda}{2} & C \end{pmatrix}. \tag{29}$$

By Sylvester's criterion, $M$is positive definite provided all its leading principal minors are positive. Consequently, there exists $c_3 > 0$such that $M$is positive definite.
Therefore,

$$\dot{V} < 0, \forall (B, N, P, F) \in \Omega \setminus \{E^*\},$$

and

$$\dot{V} = 0 \iff (B, N, P, F) = E^*.$$

Thus $V$is a strict Lyapunov function in Ω.

Theorem 4 (Global stability).
Under Assumption 1, if the coexistence equilibrium E^*exists (condition (3.3)) and the matrix $M$is positive definite, then $E^*$is globally asymptotically stable in the interior of Ω.

Proof
From the preceding analysis, $V$is positive definite and $\dot{V}$is negative definite whenever $M$is positive definite.
Hence,

$$\dot{V} < 0 \forall (B, N, P, F) \neq E^*,$$

while

$$\dot{V} = 0$$

holds only at E^*.
Therefore, $V$is a strict Lyapunov function in Ω. By Lyapunov's direct method together with LaSalle's invariance principle, the coexistence equilibrium $E^*$is globally asymptotically stable in the interior of Ω.

3.6 Parameter Values Used in Numerical Simulations

To illustrate the analytical results in Section 3, numerical simulations were conducted using the parameter values in Table 1, chosen to satisfy the model assumptions and admit a positive coexistence equilibrium.

Table 1. Parameter Values Used in Numerical Simulations

Parameter	Value	Description
(s)	0.8	Intrinsic growth rate of rare plants
(L)	100	Carrying capacity
(s_0)	0.1	Natural mortality rate
(β_1)	0.002	Human-induced depletion
(λ_1)	0.001	Population-pressure depletion
(σ_0)	0.08	Conservation rate of rare plants due to fire- and toxicant-control activity
(R)	0.5	Human growth rate
(K)	150	Human carrying capacity
(r_0)	0.1	Human depletion rate
(β)	0.001	Human–rare plant interaction rate
(λ)	0.3	Population pressure growth
(λ_0)	0.2	Population pressure decay
(Δ)	0.4	Conservation effort rate
(σ)	0.3	Conservation depletion
(δ)	0.1	Conservation reinforcement

Using these parameter values, the coexistence equilibrium was numerically obtained as

$$E* = (41.315, 132.394, 198.592, 1.174)$$

All equilibrium components are positive, confirming a biologically feasible coexistence state. The positive equilibrium value $F^* = 1.1737$indicates that conservation activities sustain rare plant persistence despite human population growth and population pressure. Furthermore, the existence condition is satisfied since

$$s + \sigma 0F* = 0.8939$$

while

$$s_0 + \beta 1N* + \lambda 1P* = 0.5634$$

Therefore,

$$0.8939 > 0.5634$$

Condition (3.3) is satisfied, confirming the existence of a positive coexistence equilibrium. Ecologically, this indicates that intrinsic plant growth and conservation efforts outweigh natural mortality, human impacts, and population pressure, allowing rare plant persistence under the selected parameter regime. Unlike classical logistic-growth models, persistence depends not only on intrinsic growth and mortality but also on the effectiveness of conservation activities.

3.7 Verification of Analytical Results

The analytical results in Section 3 were verified numerically. Since the Jacobian matrix at E^*has eigenvalues with negative real parts, the coexistence equilibrium is locally asymptotically stable, consistent with Theorem 3.

$$\lambda_1 = -0.418248$$
$$\lambda_{2,3} = -0.276791 \pm 0.127435i,$$
$$\lambda_4 = -0.200000.$$

Since all eigenvalues possess negative real parts,
$$Re(\lambda i) < 0, i = 1,2,3,4,$$

the coexistence equilibrium $E^*$is locally asymptotically stable. Therefore, the numerical results confirm the analytical stability conditions established in Theorem 3. The complex conjugate eigenvalues $\lambda_{2,3}$indicate damped transient oscillations before convergence to the coexistence equilibrium, consistent with Figure 1. The negative real parts of all eigenvalues confirm that the equilibrium remains stable, with conservation activities offsetting the destabilizing effects of human population growth and population pressure.

3.8 Numerical Simulations

The numerical simulations were carried out in Python 3 using the NumPy, SciPy, and Matplotlib libraries. The coexistence equilibrium point was computed using the nonlinear equation solver fsolve available in the SciPy optimization package, while local stability was verified through eigenvalue analysis of the Jacobian matrix using the eigvals routine.

The governing differential equations were numerically integrated using SciPy's solve_ivp with the default adaptive Runge–Kutta RK45 (Dormand–Prince) method. Simulations were performed over $t \in [0,150]$with 2000 equally spaced time points. The numerical experiments illustrate the theoretical stability results and examine the effects of conservation effort, human population pressure, and environmental disturbances on rare plant persistence. Sensitivity analyses were conducted to assess the influence of selected parameters on the long-term equilibrium. All figures were generated at 300 dpi with clear labels, legends, and line widths for improved readability.

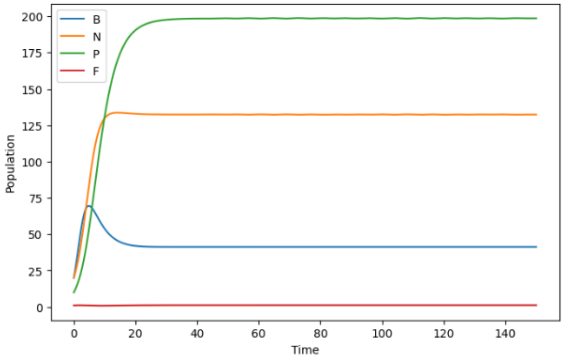

Figure 1. Temporal evolution of $B(t)$, $N(t)$, $P(t)$, and $F(t)$, showing convergence to the coexistence equilibrium $E^*$and confirming the local asymptotic stability predicted by Theorem 3.

Figure 1 shows the temporal evolution of the system variables. Despite initial transient fluctuations, all variables converge to the coexistence equilibrium, confirming its asymptotic stability.

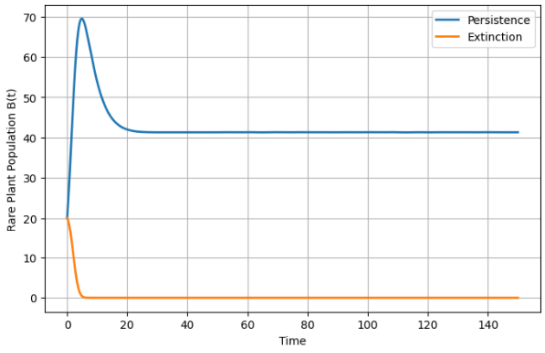

Figure 2. Persistence and extinction scenarios of the rare plant population. Adequate conservation leads to a positive equilibrium, whereas increased anthropogenic pressure and reduced conservation result in extinction.

Figure 2 illustrates two contrasting dynamical regimes. Under the baseline parameters, the rare plant population converges to the positive equilibrium ($B^* \approx 41.31$), confirming long-term persistence and the coexistence stability predicted for E_3. In contrast, removing conservation effectiveness ($\sigma_0 = 0$)and increasing anthropogenic pressure ($\beta_1\lambda_1 = 0.02$)drives the population toward extinction (1.74×10^{-6}). This result supports the theoretical prediction that persistence depends on a balance between conservation effort and disturbance intensity.

The observed persistence–extinction transition is consistent with previous mathematical ecology studies. Goshu and Endalew [3] demonstrated that insufficient conservation measures can accelerate the depletion of biological resources, while Lata et al. [8] showed that increasing human population pressure and deforestation may destabilize ecological systems and increase extinction risk. Similarly, Nababan et al. [9] reported that conservation efforts within a society-inclusive framework improve the long-term viability of rare plant populations. [7], [10], [16]. The present model supports these conclusions and further extends them by incorporating two additional ecological stressors, namely fire disturbance and toxicant exposure, within the same dynamical system.

The model indicates that rare plant persistence depends on the combined effects of toxicant exposure, conservation effort, human population pressure, and fire disturbance. Since the model is not calibrated with field data, the numerical results are illustrative rather than predictive for North Sumatra, but they support the analytical findings and provide theoretical insight into long-term population persistence.

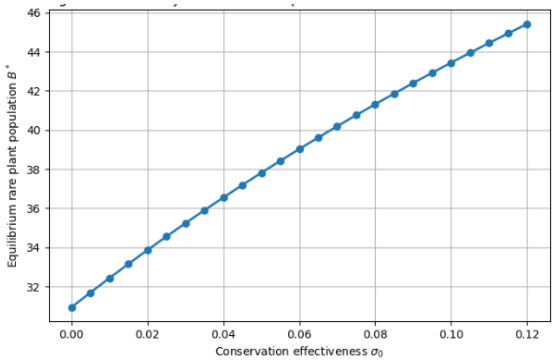

Figure 3. Sensitivity of the equilibrium population $B^*$to σ_0. Higher conservation effectiveness increases the equilibrium population, promoting long-term persistence.

Figure 3 shows the sensitivity of the equilibrium rare plant population $B^*$to changes in the conservation effectiveness parameter σ_0. The equilibrium population increases monotonically as conservation effectiveness increases. This result indicates that improvements in conservation programs can substantially enhance the long-term survival prospects of rare plant species.

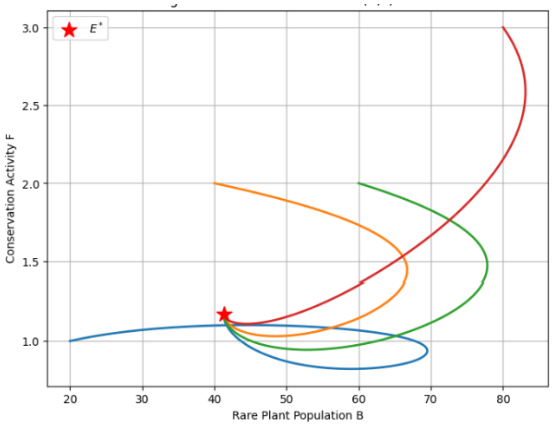

Figure 4. Phase portrait in the (B, F)-plane for different initial conditions. All trajectories approach the coexistence equilibrium E^*, illustrating the robustness of the stable equilibrium and supporting the analytical stability results.

Figure 4 depicts the phase portrait in the (B, F)-plane for several initial conditions. All trajectories converge to the same coexistence equilibrium, confirming its global stability. The negative real parts of all eigenvalues indicate that conservation activities counteract the destabilizing effects of human population growth and population pressure.

4. CONCLUSION

This study developed a nonlinear dynamical model describing interactions among rare plant populations, human populations, population pressure, conservation activities, fire disturbance, and toxicant effects. Using positivity, boundedness, equilibrium, and stability analyses, four equilibria were identified: the total-extinction (E_0), plant-extinction (E_1), human-extinction (E_2), and coexistence (E_3) equilibria. The principal mathematical contribution of this work is the derivation of explicit existence and stability conditions for the coexistence equilibrium. Using the Routh–Hurwitz criterion and Lyapunov theory, sufficient conditions for rare plant persistence under human pressure and environmental disturbances were established. The results show that persistence depends on the balance between conservation effort and interacting stressors, including population pressure, fire disturbance, and toxicant exposure. Future work will focus on calibration with ecological field data and extensions to stochastic and spatial models.

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