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Identification of Subgroups of Geometric Transformations using Linear Algebra and Group Theory

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ABSTRACT

The set of all plane geometric transformations $(\boldsymbol{\Gamma})$ forms a group under the binary operation of function composition. One of its subgroups consists of all transformations that can be expressed in the form $T(x) = Ax + v$, where A is an invertible (2x2) and v is a fixed vector $([\boldsymbol{\Gamma}])$. This study aims to identify the existence and structure of certain subgroups within $[\boldsymbol{\Gamma}]$ through a linear algebra approach. The research methods include a literature review, simulations on specific cases to obtain a more concrete understanding of the problem, and deductive reasoning based on mathematical syllogisms to derive properties and theorems that can be algebraically verified. Consistent with the research objectives, the concepts and theoretical foundations employed are drawn from the analytical properties of plane geometry and linear algebra. These concepts and theorems are revisited to ensure their relevance to the research problem and applicability in its resolution. By applying these theoretical constructs to the problem, several subgroups whose existence can be proven algebraically are identified. These subgroups include the translation subgroup, the subgroup containing rotation transformations, $[S]\cup[R]\cup[M]\cup[G]$, the group of isometries, and the group of similarities.

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1. INTRODUCTION

The topic of geometric transformations, as a course, is generally presented through a mixed approach combining axiomatic and analytic geometry. However, the proofs are typically delivered using an axiomatic geometric framework. Consequently, the material does not differ significantly from that of axiomatic geometry itself. This situation is disadvantageous for students with limited proficiency in axiomatic reasoning. They often struggle to grasp the meaning of the theorems governing geometric transformations, particularly when confronted with proof-based exercises whose complete demonstrations are not provided in the textbook. In many cases, students tend to memorize the steps of a theorem's proof without comprehending its essential meaning. Yet, when the appropriate and relevant theorems are used, such problems can be solved more easily.

On the other hand, it is well known that the set of all geometric transformations forms a group under the operation of function composition. This fact indicates that group theory can be effectively applied to characterize the set of geometric transformations as a group-structured set, both at the element level and at the level of its subsets.

Furthermore, consider the function

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix}$$

for every x∈$\mathbf{R}^2$, where det(A)≠0, (A) is a 2x2 real matrix, and $\begin{pmatrix} m \\ n \end{pmatrix}$ is a fixed element of $\mathbf{R}^2$. This bijective mapping from $\mathbf{R}^2$ to $\mathbf{R}^2$ shows that problems involving geometric transformations can be discussed within the framework of **linear algebra**, particularly through properties related to systems of two-variable linear equations.

Moreover, since this study is conducted on a set with well-defined membership properties, the derived results become richer and more detailed, thereby yielding outcomes with higher practical value. In this sense, they can be used to solve planar transformation problems more effectively. Consequently, the findings can be utilized to enrich the content of *Geometric Transformations* and *Algebraic Structures* courses, providing a wider variety of instructional material. Specifically, in the *Algebraic Structures* course—comprising *Group Theory* and *Ring Theory*—if the research results are employed as illustrative examples in teaching, the abstractness of the concepts and theorems can be reduced. Thus, the outcomes of this study help lower the level of abstraction in algebra courses by providing tangible geometric interpretations.

The objectives of this research are to determine the following properties of [**Γ**]:

a. [**Γ**] forms a group;
b. [**Γ**] possesses proper subgroups; and
c. [**Γ**] contains normal subgroups.

Geometric interpretations of these properties are subsequently provided, thereby reducing the abstraction of **Γ** as a group and enabling analytical and algebraic exploration of [**Γ**].

The significance of this research lies in the development of mathematical knowledge, particularly in applying group theory and linear algebra to problems of planar geometric transformations. This allows one to identify their characteristics both at the element level and in the overall system structure, while simultaneously reducing the level of abstraction inherent in the traditional axiomatic-geometric treatment toward a more analytical perspective.

Symbols and Notation

$$\boldsymbol{R}x\boldsymbol{R} = \{(x,\mathrm{y})|x,\, \mathrm{y} \in \boldsymbol{R}\};\ \boldsymbol{R}^2 = \left\{\begin{pmatrix} x \\ y \end{pmatrix} |x, \mathrm{y} \in \boldsymbol{R}\right\}.$$

It is clear that **R**x**R** and $\mathbf{R}^2$ differ only symboliccally. Therefore, the function T : **R**x**R** → **R**x**R** is henceforth written simply as T : $\mathbf{R}^2 \to \mathbf{R}^2$ for simplicity. The identity transformation T_0: $\mathbf{R}^2 \to \mathbf{R}^2$ is defined by $T_0(\mathbf{x})=x$ for all x∈$\mathbf{R}^2$.
Let

$$M_{2x2} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0; a, b, c, d \in \boldsymbol{R} \right\}.$$

A vector v∈$\mathbf{R}^2$ can be represented as $\overrightarrow{\mathrm{OP}}$ from the origin O=(0,0) to a point P=(a,b).
Define the following sets:

$$\Gamma = \{T\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2 | T \text{ is bijective}\},$$

$$[\Gamma] = \{T_{\mathrm{A,v}}\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2 \mid \forall \mathrm{x} \in \boldsymbol{R}^2 \ni T_{\mathrm{A,v}}(\mathrm{x}) = \mathrm{Ax} + \mathrm{v};\ \mathrm{v} \in \boldsymbol{R}^2;\ \det(\mathrm{A}) \neq 0;\ \mathrm{A} \in \mathrm{M}_{2x2}\},$$

$$[\boldsymbol{RS}] = \left\{T_r\colon \mathrm{R}^2 \to \mathrm{R}^2 \middle| \mid \forall \mathrm{x} \in \mathbf{R}^2 \ni T_r(\mathrm{x}) = \mathrm{A_r x} + \mathrm{v};\ \mathrm{v} \in \mathbf{R}^2;\ A_r = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \theta \in [0, 2\pi]\right\},$$

$$[\boldsymbol{G}] = \left\{ \mathrm{M}: \mathbf{R}^2 \to \mathbf{R}^2 \,\middle|\, \begin{array}{c} \forall \mathrm{x}\in\mathbf{R}^2 \ni \mathrm{G(x)} = A_m\mathrm{x} + \mathrm{v} + v_t;\ A_\mathrm{m} = \begin{pmatrix} -\cos2\theta & -\sin2\theta \\ -\sin2\theta & \cos2\theta \end{pmatrix};\ \mathrm{v} = 2\mathrm{p}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}; \\ v_\alpha = \alpha\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix};\ \theta\in[0,2\pi];\ \mathrm{p}\geq 0;\ \mathrm{p},\alpha\in\mathbf{R} \end{array} \right\},$$

$$[\boldsymbol{G}]^{-1} = \{\mathrm{T}\in[\Gamma] | \mathrm{T}G = \mathrm{GT} = \mathrm{I}; \mathrm{G}\in[\mathbf{G}]\}.$$

The set of translation is

$$[\boldsymbol{S}] = \{\mathrm{S}_v: \mathbf{R}^2 \to \mathbf{R}^2 | \forall \mathrm{x}\in\mathbf{R}^2 \ni S_v(x) = \mathrm{x} + \mathrm{v};\ \mathrm{v}\in\mathbf{R}^2\},$$

with inverse set

$$[\boldsymbol{S}]^{-1} = \{\mathrm{T}\in[\Gamma] | \mathrm{TS} = \mathrm{ST} = \mathrm{I}; \mathrm{S}\in[\mathbf{S}]\}.$$

The set of reflection is

$$[\boldsymbol{M}] = \left\{ \mathrm{M}: \mathbf{R}^2 \to \mathbf{R}^2 \,\middle|\, \begin{array}{c} \forall \mathrm{x}\in\mathbf{R}^2 \ni \mathrm{M(x)} = A_m\mathrm{x} + \mathrm{v};\ A_\mathrm{m} = \begin{pmatrix} -\cos2\theta & -\sin2\theta \\ -\sin2\theta & \cos2\theta \end{pmatrix};\ \mathrm{v} = 2\mathrm{p}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}; \\ (\theta, \mathrm{p})\in[0,2\pi]\mathrm{x}\mathbf{R};\ \mathrm{p}\geq 0 \end{array} \right\},$$

with inverse set

$$[\boldsymbol{M}]^{-1} = \{\mathrm{T}\in[\Gamma] | \mathrm{TM} = \mathrm{MT} = \mathrm{I}; \mathrm{M}\in[\mathbf{M}]\}.$$

Since reflection are involutory, [M] is not closed under composition and hence does not form a group. The set of rotation is

$$[\boldsymbol{R}] = \left\{ R_{A,v}: \mathbf{R}^2 \to \mathbf{R}^2 \,\middle|\, \begin{array}{c} \forall \mathrm{x}\in\mathbf{R}^2 \ni R_{A,v}(\mathrm{x}) = A_r(x - v) + \mathrm{v};\ \mathrm{v} = \overrightarrow{OP};\ A_r = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \\ \mathrm{P} = (\mathrm{a}, \mathrm{b});\ \theta\in[0,2\pi] \end{array} \right\},$$

with inverse set

$$[\boldsymbol{R}]^{-1} = \{\mathrm{T}\in[\Gamma] | \mathrm{TR} = \mathrm{RT} = \mathrm{I}; \mathrm{R}\in[\mathbf{R}]\}.$$

Let [$\mathbf{R}_0$] denote the subset of rotation about the origin O=(0,0); it can be shown that [$\mathbf{R}_0$] forms a group. The set of isometries is

$$[\boldsymbol{\Gamma}_{\mathrm{iso}}] = \left\{ \mathrm{T}: \mathbf{R}^2 \to \mathbf{R}^2 \middle| T(\mathrm{x}) = \mathrm{Ax} + \mathrm{v}_0; \mathrm{v}_0 \in \mathbf{R}^2; A = \begin{pmatrix} a & b \\ \mp b & \pm a \end{pmatrix};\ \mathrm{a}^2 + \mathrm{b}^2 = 1 \right\},$$

while its subgroup of isometries fixing the origin is

$$[\boldsymbol{\Gamma}_{\mathrm{iso:0}}] = \left\{ \mathrm{T}: \mathbf{R}^2 \to \mathbf{R}^2 \middle| T(\mathrm{x}) = \mathrm{Ax} + \mathrm{v}_0; \mathrm{v}_0 \in \mathbf{R}^2; A = \begin{pmatrix} a & b \\ \mp b & \pm a \end{pmatrix};\ \mathrm{a}^2 + \mathrm{b}^2 = 1 \right\},$$

Similarly, the set of similarities is

$$[\boldsymbol{\Gamma}_{\mathrm{s}}] = \left\{ \mathrm{T}: \mathbf{R}^2 \to \mathbf{R}^2 \middle| T(\mathrm{x}) = \mathrm{Ax} + \mathrm{v}_0; \mathrm{v}_0 \in \mathbf{R}^2; A = \begin{pmatrix} a & b \\ \mp b & \pm a \end{pmatrix};\ \mathrm{a}^2 + \mathrm{b}^2 \neq 0 \right\},$$

and its subgroup fixing the origin is

$$[\boldsymbol{\Gamma}_{\mathrm{s:0}}] = \left\{ \mathrm{T}: \mathbf{R}^2 \to \mathbf{R}^2 \middle| T(\mathrm{x}) = \mathrm{Ax};\ A = \begin{pmatrix} a & b \\ \mp b & \pm a \end{pmatrix};\ \mathrm{a}^2 + \mathrm{b}^2 \neq 0 \right\}.$$

2. RESEARCH METHOD

In accordance with the deductive nature of mathematics as a discipline, the research methodology employed in this study consists of literature review, simulation of specific cases to obtain a more concrete understanding of the problems under consideration, and deductive reasoning based on the laws of mathematical syllogism. The results of the study—comprising definitions, properties, and theorems—were obtained by collecting relevant concepts and established theorems, examining their interrelations and applicability to the research problem, and deriving new conclusions grounded on these foundations.

In this research, both the concepts and theorems revisited include those pertaining to group theory, analytic geometry in the Euclidean plane, and linear algebra insofar as they are relevant to the investigated problem.

Specifically, the group-theoretic notions and theorems reconsidered are the definitions and properties of groups, subgroups, and normal subgroups, together with the corresponding fundamental theorems. Meanwhile, the concepts and theorems of linear algebra revisited are those related to systems of two-variable linear equations. This shift in the methodological approach to the study of the set of geometric transformations provides new perspectives on research topics involving geometric transformations in $\mathbf{R}^2$, both in terms of structural composition and the distinctive properties of each transformation.

The findings from the literature review can be summarized as follows. The set **Γ**, equipped with a binary operation (composition of functions), forms a group, and **[Γ]** constitutes a subgroup of **Γ**. Since **[Γ]** is a specific subset, the application of group-theoretic principles to **Γ** yields properties that are characteristic of this structure. In other words, the characteristics of **Γ** cannot yet be fully identified in a complete and analytical manner; the level of completeness and detail achievable depends largely on the depth of the analytical process undertaken in the research.

The group-theoretic definitions and properties employed to analyze the research problem are as follows:

a. A nonempty subset H of a group G is a subgroup of (G) if and only if for every x, y ∈ H, the element xy^{-1} also belongs to H [1], [2].
b. A subgroup (N) of (G) is a normal subgroup if and only if for every g ∈ G and x ∈ N, it follows that gxg^{-1} ∈ N) [1], [2], [3], [4].

These theorems were successively employed to determine whether a subset of **[Γ]** qualifies as a subgroup, and subsequently to verify whether a given subgroup is normal.

From the perspective of linear algebra, the following theorems are utilized:

- If A is a 2x2 real matrix and b is a 2x1 real matrix, then the following statements are equivalent:
    1. A is invertible;
    2. det(A) ≠ 0;
    3. the system of linear equations A x = b has a unique solution [5].
- The determinant of the product of two 2x2 real matrices satisfies

$$\text{det(AB)}\det(AB) = \det(A)\det(B),$$

where (A) and (B) are real (2\times2) matrices [5], [6].

Through the combination of these group-theoretic and linear-algebraic frameworks, the research establishes a deductive foundation for characterizing the structural and transformational properties of **[Γ]** as a subgroup within the geometric transformation group of $\mathbf{R}^2$.

3. RESULT AND ANALYSIS

In accordance with the scope of the research topic, the innovative contribution of this study lies in the methodological shift in the treatment of geometric transformations. The discussion, which traditionally relied on the concepts and theorems of axiomatic and analytic geometry, has been reformulated through the frameworks of algebraic structures and linear algebra.

This reformulation not only simplifies the analytical process but also broadens the scope of investigation, allowing for the characterization of geometric transformations in $\mathbf{R}^3$ and, more generally, in $\mathbf{R}^n$ for any natural number n. Furthermore, the results demonstrate that the study of geometric transformations can be conducted entirely within an algebraic framework, provided that the scope of discussion is properly constrained.

Results of Special Matrix Multiplications

The following are the results of matrix multiplications involving special forms that are employed in the process of proving the theorems established in this study.

Theorem 3.1

Let

$$\mathrm{A} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \text{ dan } B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}, \text{ where } \alpha,\beta \in [0,2\pi].$$

Then

$$AB = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}.$$

Proof

Multiply:

$$AB = \begin{pmatrix} \cos\alpha\cos\beta\text{-}\sin\alpha\sin\beta & -\cos\alpha\sin\beta\text{-}\sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \text{-}\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}.$$

Using the angle-addition identities

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,\ \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

we obtain

$$AB = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}.$$

Theorem 3.2
Let

$$A = \begin{pmatrix} \text{-}\cos 2\alpha & -\sin 2\alpha \\ \text{-}\sin 2\alpha & \cos 2\alpha \end{pmatrix}, B = \begin{pmatrix} \text{-}\cos 2\beta & -\sin 2\beta \\ \text{-}\sin 2\beta & \cos 2\beta \end{pmatrix} \alpha, \beta \in [0, 2\pi].$$

Then

$$AB = \begin{pmatrix} \cos 2(\alpha\text{-}\beta) & -\sin 2(\alpha\text{-}\beta) \\ \sin 2(\alpha\text{-}\beta) & \cos 2(\alpha\text{-}\beta) \end{pmatrix}, A^2 = I.$$

Proof
Compute

$$AB = \begin{pmatrix} \text{-}\cos 2\alpha & -\sin 2\alpha \\ \text{-}\sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} \text{-}\cos 2\beta & -\sin 2\beta \\ \text{-}\sin 2\beta & \cos 2\beta \end{pmatrix}.$$

By matrix multiplication,

$$AB = \begin{pmatrix} (\cos 2\alpha)\cos 2\beta + (\sin 2\alpha)\sin 2\beta & (\cos 2\alpha)(\sin 2\beta) - (\sin 2\alpha)\cos 2\beta \\ (\sin 2\alpha)\cos 2\beta - (\cos 2\alpha)\sin 2\beta & (\sin 2\alpha)\sin 2\beta + (\cos 2\alpha)\cos 2\beta \end{pmatrix}$$

Applying the trigonometric identities
cos (x-y) = cos x cos y + sin x siny and sin (x-y) = sin x cos y - cos x siny,
we obtain

$$AB = \begin{pmatrix} \cos 2\,(\alpha\text{-}\beta) & -\sin 2\,(\alpha\text{-}\beta) \\ \sin 2\,(\alpha\text{-}\beta) & \cos 2\,(\alpha\text{-}\beta) \end{pmatrix}.$$

If A=B, then α=β and hence

$$A^2 = AB = \begin{pmatrix} \cos 2(0) & -\sin 2(0) \\ \sin 2(0) & \cos 2(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Theorem 3.3
Let

$$A = \begin{pmatrix} \text{-}\cos 2\alpha & -\sin 2\alpha \\ \text{-}\sin 2\alpha & \cos 2\alpha \end{pmatrix}, B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}, \alpha, \beta \in [0, 2\pi].$$

Then

$$AB = \begin{pmatrix} \text{-}\cos(2\alpha\text{-}\beta) & -\sin(2\alpha\text{-}\beta) \\ \text{-}\sin(2\alpha\text{-}\beta) & \cos(2\alpha\text{-}\beta) \end{pmatrix}, \quad BA = \begin{pmatrix} \text{-}\cos(\beta\text{-}2\alpha) & -\sin(\beta\text{-}2\alpha) \\ \text{-}\sin(\beta\text{-}2\alpha) & \cos(\beta\text{-}2\alpha) \end{pmatrix}.$$

Moreover, if β=2α, then

$$AB = BA = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof
Compute

$$AB = \begin{pmatrix} \text{-}\cos 2\alpha & -\sin 2\alpha \\ \text{-}\sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

By direct multiplication,

$$AB = \begin{pmatrix} (\text{-}\cos 2\alpha)\cos\beta + (\text{-}\sin 2\alpha)\sin\beta & (\text{-}\cos 2\alpha)(\text{-}\sin\beta) + (\text{-}\sin 2\alpha)\cos\beta \\ (\text{-}\sin 2\alpha)\cos\beta + (\cos 2\alpha)\sin\beta & (\text{-}\sin 2\alpha)(\text{-}\sin 2\alpha) + (\cos 2\alpha)\cos\beta \end{pmatrix}.$$

Type equation here.

Simplifying and applying trigonometric identities yileds

$$AB = \begin{pmatrix} -\cos(2\alpha\text{-}\beta) & -\sin(2\alpha\text{-}\beta) \\ -\sin(2\alpha\text{-}\beta) & \cos(2\alpha\text{-}\beta) \end{pmatrix}.$$

Similarly,

$$BA = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \text{-}\cos 2\alpha & -\sin 2\alpha \\ \text{-}\sin 2\alpha & \cos 2\alpha \end{pmatrix},$$

which leads to

$$BA = \begin{pmatrix} -cos(\beta\text{-}2\alpha) & -sin(\beta\text{-}2\alpha) \\ -sin(\beta\text{-}2\alpha) & cos(\beta\text{-}2\alpha) \end{pmatrix}.$$

Finally, when β=2α,

$$\mathrm{AB} = \mathrm{BA}\begin{pmatrix} \text{-}\cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorema 3.4
Let

$$A = \begin{pmatrix} a & b \\ \mp b & \pm a \end{pmatrix}, B = \begin{pmatrix} m & n \\ \mp n & \pm m \end{pmatrix},$$

where $a^2+b^2=1$; $m^2+n^2=1$; a,b,m,n∈**R.**
Then

$$AB = \begin{pmatrix} \mathrm{am} \mp \mathrm{bn} & \mathrm{an} \pm \mathrm{bm} \\ \mp(\mathrm{an} \pm \mathrm{bm}) & \pm(\mathrm{am} \mp \mathrm{bn}) \end{pmatrix};\ (\mathrm{am} \mp \mathrm{bn})^2 + (\mathrm{an} \pm \mathrm{bm})^2 = 1.$$

Proof
By direct multiplication,

$$AB = \begin{pmatrix} a & b \\ \mp b & \pm a \end{pmatrix}\begin{pmatrix} m & n \\ \mp n & \pm m \end{pmatrix} = \begin{pmatrix} \mathrm{am} \mp \mathrm{bn} & \mathrm{an} \pm \mathrm{bm} \\ \mp\mathrm{bm\text{-}an} & \mp\mathrm{bn} + \mathrm{am} \end{pmatrix},$$

Since det(A)=1 and det(B)=1, it follows that

$$de(AB) = \det(A)\det(B) = 1. \square$$

Hence,

$$\left|\begin{pmatrix} \mathrm{Am} \mp \mathrm{bn} & \mathrm{an} \pm \mathrm{bm} \\ \mp(\mathrm{an} \pm \mathrm{bm}) & \pm(\mathrm{am} \mp \mathrm{bn}) \end{pmatrix}\right| = 1,$$

and consequently $(\mathrm{am} \mp \mathrm{bn})^2 + (\mathrm{an} \pm \mathrm{bm})^2 = 1$.

Theorem 3.5
Let

$$A = \begin{pmatrix} a & b \\ \mp b & \pm a \end{pmatrix}, B = \begin{pmatrix} m & n \\ \mp n & \pm m \end{pmatrix},$$

where $a^2+b^2\neq0$; $m^2+n^2\neq0$; a,b,m,n∈**R**.
Then

$$AB = \begin{pmatrix} \mathrm{am} \mp \mathrm{bn} & \mathrm{an} \pm \mathrm{bm} \\ \mp(\mathrm{an} \pm \mathrm{bm}) & \pm(\mathrm{am} \mp \mathrm{bn}) \end{pmatrix};\ (\mathrm{am} \mp \mathrm{bn})^2 + (\mathrm{an} \pm \mathrm{bm})^2 \neq 0.$$

Proof
By direct multiplication,

$$AB = \begin{pmatrix} a & b \\ \mp b & \pm a \end{pmatrix}\begin{pmatrix} m & n \\ \mp n & \pm m \end{pmatrix} = \begin{pmatrix} \mathrm{am} \mp \mathrm{bn} & \mathrm{an} \pm \mathrm{bm} \\ \mp\mathrm{bm\text{-}an} & \mp\mathrm{bn} + \mathrm{am} \end{pmatrix},$$

Because det(A)≠0 and det(B) ≠0,

$$de(AB) = \det(A)\det(B) \neq 0.$$

Therefore,

$$(\mathrm{am} \mp \mathrm{bn})^2 + (\mathrm{an} \pm \mathrm{bm})^2 = \left|\begin{pmatrix} \mathrm{Am} \mp \mathrm{bn} & \mathrm{an} \pm \mathrm{bm} \\ \mp(\mathrm{an} \pm \mathrm{bm}) & \pm(\mathrm{am} \mp \mathrm{bn}) \end{pmatrix}\right| \neq 0. \square$$

The Inverse of [S], [M], [R], dan [G]

In general, it can be shown that the inverse of a translation is also a translation, the inverse of a reflection is a reflection, the inverse of a rotation is a rotation, and the inverse of a glide reflection is a glide reflection.

Theorem 3.6

If $T\in[\mathbf{S}]$, then $T^{-1}\in[\mathbf{S}]$, hence $[\boldsymbol{S}]^{-1} = [\boldsymbol{S}]$.

Proof.

Let $T\in[\mathbf{S}]$. Since $T\in[\mathbf{S}]$, then for all $x \in \mathbf{R}^2$,

$$T(x) = \mathrm{x} + \mathrm{v};\ \mathrm{v}\in\mathbf{R}^2,$$

so that $TT^{-1} = T^{-1}T = I$ with

$$T^{-1}(x) = \mathrm{x} + (-v);\ -\mathrm{v}\in\mathbf{R}^2.$$

Therefore, $T^{-1} \in[\boldsymbol{S}]$, and thus $[\boldsymbol{S}]^{-1}\subset[\boldsymbol{S}]$.
Assume $[\boldsymbol{S}]^{-1}\neq[\boldsymbol{S}]$. Consequently, there exists $T_1 \in[\boldsymbol{S}]$ such that $T_1 \notin[\boldsymbol{S}]^{-1}$.
Since $T_1 \notin[\boldsymbol{S}]^{-1}$, for every $\mathrm{W}\in[\boldsymbol{\Gamma}]$ we have $T_1\mathrm{W}\neq\mathrm{W}T_1$ or $T_1\mathrm{W}\neq\mathrm{I}$.
This means that T_1 does not have an inverse in $[\boldsymbol{\Gamma}]$, which contradicts the fact that $[\boldsymbol{\Gamma}]$ forms a group.
Hence, $[\boldsymbol{S}]^{-1} = [\boldsymbol{S}]$. □

Theorem 3.7

If $T\in[\mathbf{M}]$, then $T^{-1}\in[\mathbf{M}]$, hence $[\boldsymbol{M}]^{-1} = [\boldsymbol{M}]$.

Proof.

Let $T\in[\mathbf{M}]$. Since every reflection is an involution, $T^2 = \mathrm{I} \Leftrightarrow \mathrm{T} = T^{-1}$.
Because $T = T^{-1}$ for all $\mathrm{T} \in [\mathbf{M}]$, it follows that $[\boldsymbol{M}]^{-1} = [\boldsymbol{M}]$. □

Theorem 3.8

If $T\in[\mathbf{R}]$, then $T^{-1}\in[\mathbf{R}]$, hence $[\boldsymbol{R}]^{-1} = [\boldsymbol{R}]$.

Proof.

Let $T\in[\mathbf{R}]$. Since $T\in[\mathbf{R}]$, for every $\mathrm{x}\in\mathbf{R}^2$ we have

$$\mathrm{T(x)} = A_r(x - v) + \mathrm{v},$$

where

$$\mathrm{v} = \overrightarrow{OP};\ A_r = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix};\ \mathrm{P} = (\mathrm{a}, \mathrm{b});\ \theta\in[0,2\pi].$$

Then

$$TT^{-1} = T^{-1}T = I,$$

with

$$\forall \mathrm{x}\in\mathbf{R}^2 \ni T^{-1}(x) = A_r{}^{-1}(x - v)\mathrm{x} + \mathrm{v};\ A_r{}^{-1} = \begin{pmatrix} \cos(2\pi - \theta) & -\sin(2\pi - \theta) \\ \sin(2\pi - \theta) & \cos(2\pi - \theta) \end{pmatrix}.$$

Hence, $T^{-1} \in[\boldsymbol{R}]$, implying $[\boldsymbol{R}]^{-1}\subset[\boldsymbol{R}]$.
Assume $[\boldsymbol{R}]^{-1}\neq[\boldsymbol{R}]$. Then there exists $T_2 \in[\boldsymbol{R}]$ such that $T_2 \notin[\boldsymbol{R}]^{-1} \Leftrightarrow \forall \mathrm{U}\in[\boldsymbol{\Gamma}]\ni T_2\mathrm{U}\neq\mathrm{U}T_2 \vee T_2\mathrm{U}\neq\mathrm{I}$.
This means that T_2 does not have an inverse in $[\boldsymbol{\Gamma}]$, which contradicts the fact that $[\boldsymbol{\Gamma}]$ is a group.
Therefore, $[\boldsymbol{R}]^{-1} = [\boldsymbol{R}]$. □

Theorem 3.9

If $T\in[\mathbf{G}]$, then $T^{-1}\in[\mathbf{G}]$; hence $[\boldsymbol{G}]^{-1} = [\boldsymbol{G}]$.

Proof.

Let $T\in[\mathbf{G}]$. Since $T\in[\mathbf{G}]$, we may write $T=MS_v$, where $S_v\in[\mathbf{S}]$, $M_g\in[\mathbf{M}]$, g is reflection axis of M_g, and the direction vector is parallel to the line g.
Then we have

$$\mathrm{T}(\mathrm{S}_{-v}\mathrm{M}) = (\mathrm{MS}_v)(\mathrm{S}_{-v}\mathrm{M}) = \mathrm{MIM} = \mathrm{M}^2 = \mathrm{I}.$$

This implies that $T^{-1}=S_{-v}M\in[\mathbf{G}]$. Therefore, $[\boldsymbol{G}]^{-1}\subset[\boldsymbol{G}]$.
Assume that $[\boldsymbol{G}]^{-1}\neq[\boldsymbol{G}]$. Then there exists $T_3 \in[\boldsymbol{G}]$ such that $T_0 \notin[\boldsymbol{G}]^{-1}$.
Since $T_3 \notin[\boldsymbol{G}]^{-1}$, for every $\mathrm{V}\in[\boldsymbol{\Gamma}]$ we have $T_3\mathrm{V}\neq\mathrm{V}T_3$ or $T_3\mathrm{V}\neq\mathrm{I}$.
Hence T_3 has no inverse in $[\boldsymbol{\Gamma}]$, contradicting the fact that $[\boldsymbol{\Gamma}]$ is a group.
Therefore, $[\boldsymbol{G}]^{-1} = [\boldsymbol{G}]$. □

Several Subgroups of $[\boldsymbol{\Gamma}]$

Before discussing the intrinsic properties of the subgroups of $[\boldsymbol{\Gamma}]$ in depth, it is useful to first recognize several particular characteristics regarding the membership of $[\boldsymbol{\Gamma}]$ and its easily identifiable subgroups—both from journal articles and standard textbooks—so that the structure of $[\boldsymbol{\Gamma}]$ can be better understood.

The following statements hold in the group [**Γ**]:

a. The set of all translations, denoted by [S], forms a normal subgroup of [**Γ**].
b. The set of all rotations about the origin, denoted by [$\mathbf{R_0}$], forms a subgroup of [**Γ**].
c. Since the set of all rotations is not closed under composition, it does not form a group (Ikrar, 2024).

The following theorem shows the existence of a subgroup within [**Γ**] that contains [**R**].

Theorem 3.10
The set [**RS**] forms a group and contains [**R**].

Proof.
Let $R_{A,v}$ and $R_{B,w}$ be elements of [**R**]. Since $R_{A,v}$, $R_{B,w} \in$ [**R**], we have

$$\mathrm{R}_{A,v}(\mathrm{x}) = A_{r1}x + \mathrm{v}_1, \mathrm{R}_{B,w}(\mathrm{x}) = A_{r2}x + \mathrm{v}_2$$

where $\mathrm{v}_1 = \overrightarrow{OP_1}$, $\mathrm{v}_2 = \overrightarrow{OP_2} \in \mathbf{R}^2$,

$$A_{r1} = \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}, A_{r2} = \begin{pmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{pmatrix},$$

with $P_1 = (a_1, b_1)$, $P_2 = (a_2, b_2)$; $\theta_1, \theta_2 \in [0, 2\pi]$.

The inverse of $\mathbf{T}_2$ is given by

$$T_2^{-1}(\mathrm{x}) = \begin{pmatrix} \cos(-\theta_2) & -\sin(-\theta_2) \\ \sin(-\theta_2) & \cos(-\theta_2) \end{pmatrix} (x - v_2),$$

since

$$A_{r2}{}^{-1} = \begin{pmatrix} \cos \theta_2 & \sin \theta_2 \\ -\sin \theta_2 & \cos \theta_2 \end{pmatrix} = \begin{pmatrix} \cos(-\theta_2) & -\sin(-\theta_2) \\ \sin(-\theta_2) & \cos(-\theta_2) \end{pmatrix}.$$

Then

$$A_{r1}A_{r2}{}^{-1} = \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix}.$$

Hence,

$$T_1 T_2^{-1}(x) = A_{r1}A_{r2}{}^{-1}\bar{x} - A_{r1}A_{r2}{}^{-1}v_2 + v_1 = \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix} \bar{x} + v_3$$

where $v_3 = -A_{r1}A_{r2}{}^{-1}v_2 + v_1$.
Let $R_{A,v}$ be a rotation about a point P with rotation angle θ. Then for all $x \in \mathbf{R}^2$,

$$R_{A,v}(\mathrm{x}) = A_r(x - v) + \mathrm{v},$$

where $\mathrm{v} = \overrightarrow{OP}$,

$$A_r = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \mathrm{P} = (\mathrm{a}, \mathrm{b}), \theta \in [0, 2\pi].$$

Since

$$R_{A,v}(\mathrm{x}) = A_r(x - v)\mathrm{x} + \mathrm{v} = A_r x + -A_r v + \mathrm{v} = A_r x + v_1,$$

with $v_1 = -A_r v + \mathrm{v} \in \mathbf{R}^2$, it follows that $R_{A,v} \in$ [**RS**]. □

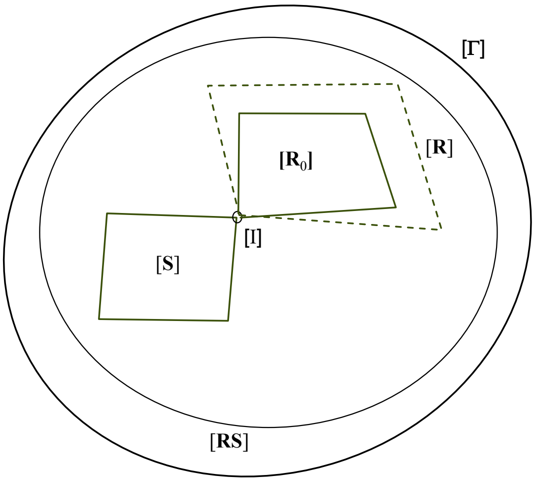

Figure 1. *Venn diagram illustrating the subgroups contained in [RS].*

Theorem 3.11

The union $[\mathbf{S}]\cup[\mathbf{R}]\cup[\mathbf{M}]\cup[\mathbf{G}]$ forms a group within $[\boldsymbol{\Gamma}]$.

Proof.

The set $[\mathbf{S}]\cup[\mathbf{R}]\cup[\mathbf{M}]\cup[\mathbf{G}]$ is closed under composition [7], as shown in Table 1.

Table 1. Composition of the Four Basic Transformations

O	S	M	R	G
S	S	G∨M	R	G∨M
M	G∨M	R∨S	G	R∨S
R	R	G	R∨S	G∨M
G	G∨M	R∨S	G∨M	R∨S

Furthermore, the inverse of every element in $[\mathbf{S}]\cup[\mathbf{R}]\cup[\mathbf{M}]\cup[\mathbf{G}]$ also belongs to the same set. That is,

$$[\mathbf{S}]\cup[\mathbf{R}]\cup[\mathbf{M}]\cup[\mathbf{G}])^{-1}\subset \mathbf{S}]\cup[\mathbf{R}]\cup[\mathbf{M}]\cup[\mathbf{G}].$$

Since $[\mathbf{S}]^{-1}=[\mathbf{S}]$, $[\mathbf{R}]^{-1}=[\mathbf{R}]$, $[\mathbf{M}]^{-1}=[\mathbf{M}]$, and $[\mathbf{G}]^{-1}=[\mathbf{G}]$, we obtain

$$[\mathbf{S}]\cup[\mathbf{R}]\cup[\mathbf{M}]\cup[\mathbf{G}])^{-1} = [\mathbf{S}]^{-1}\cup[\mathbf{R}]^{-1}\cup[\mathbf{M}]^{-1}\cup[\mathbf{G}]^{-1} = [\ \mathbf{S}]\cup[\mathbf{R}]\cup[\mathbf{M}]\cup[\mathbf{G}].$$

Hence, the set forms a group in $[\boldsymbol{\Gamma}]$.

$([\mathbf{S}]\cup[\mathbf{R}]\cup[\mathbf{M}]\cup[\mathbf{G}])^{-1}= [\mathbf{S}]^{-1}\cup[\mathbf{R}]^{-1}\cup[\mathbf{M}]^{-1}\cup[\mathbf{G}]^{-1}=[\mathbf{S}]\cup[\mathbf{R}]\cup[\mathbf{M}]\cup[\mathbf{G}]$. □

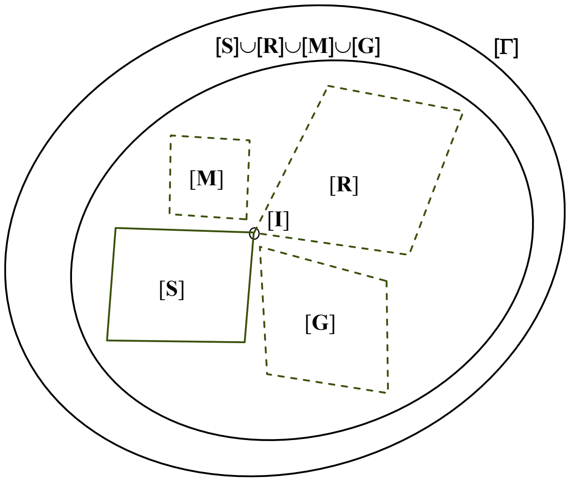

Figure 2. *Venn diagram illustrating the subgroups in [Γ] containing [S], [R], [M], and [G]*

Theorem 3.12

The sets $[\boldsymbol{\Gamma}_{iso}]$, $[\boldsymbol{\Gamma}_{iso:0}]$, $[\boldsymbol{\Gamma}_{s}]$, $[\boldsymbol{\Gamma}_{s:0}]$, dan $[\boldsymbol{\Gamma}]$ membentuk grup dalam $\boldsymbol{\Gamma}$.

Proof.

From the formulas of isometric and similarity transformations, and by applying Theorems 4 and 5 regarding matrix multiplication, it follows directly that $[\boldsymbol{\Gamma}_{iso}]$, $[\boldsymbol{\Gamma}_{iso:0}]$, $[\boldsymbol{\Gamma}_{s}]$, dan $[\boldsymbol{\Gamma}_{s:0}]$ form groups within $[\Gamma]$ [7]. □

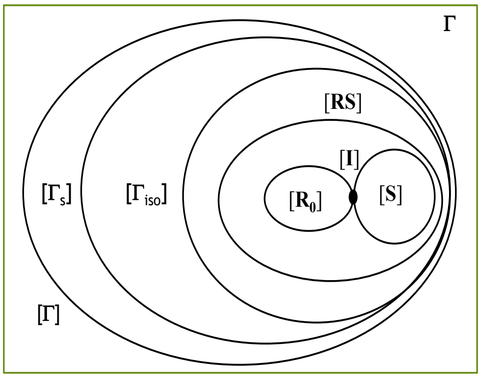

Figure 3. *Venn diagram showing the subgroups of [Γ] containing* $[\mathbf{S}]$, $[\mathbf{R}_0]$, $[\mathbf{RS}]$, $[\boldsymbol{\Gamma}_{iso}]$, dan $[\boldsymbol{\Gamma}_{s}]$

4. CONCLUSION

The results of this study show that subgroups of plane geometric transformations can be accurately identified through an algebraic approach. Therefore, further and more detailed research can still be conducted to obtain comprehensive information regarding the existence and classification of subgroups within the group of geometric transformations.

The implementation of these findings in real-world problems has not been discussed in this study and remains an open question for future investigation. Based on the experience during this research process, it is still possible to discover additional subgroups with special structures and practical significance.

The main conclusion is that geometric transformations on the Euclidean plane can be completely analyzed algebraically when the universe of discourse is restricted to $[\Gamma]$.

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