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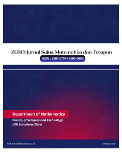

Routh-Hurwitz Stability Analysis of the Predator-Prey Model with Prey Population Harvesting in Polluted Aquatic Ecosystems

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ABSTRACT

Environmental pollution and overharvesting are critical external factors that disrupt predator–prey balance in aquatic ecosystems. This study develops a two-dimensional nonlinear predator–prey model incorporating both toxicity and harvesting. Local stability is analyzed using the Routh–Hurwitz criterion, and findings are validated through numerical simulations under varied initial conditions. The system yields four equilibria: E0, E1 and E1 are unstable extinction states, while the interior equilibrium E^*= (0.4146, 1.0899) is locally stable, with Tr(J)=-1.3052 and det(J)=0.4177. Stability is preserved as long as the combined toxicity–harvesting parameter remains below approximately 4.1-4.2 day-1. The novelty of this work lies in explicitly quantifying threshold effects of harvesting and toxicity, showing that coexistence is achievable under moderate external pressures. These results highlight that sustainable management requires keeping exploitation and pollution below critical thresholds to ensure long-term persistence of both prey and predator.

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1. INTRODUCTION

Aquatic ecosystems play a vital role in regulating nutrient cycles, supporting fisheries, and sustaining human livelihoods. However, the water quality of many such systems has deteriorated sharply in recent decades because of increasing inputs of industrial, mining, and agricultural waste. Elevated concentrations of heavy metals and synthetic chemicals directly impair aquatic life, particularly prey organisms whose photosynthesis, growth, and reproduction are sensitive to pollutant stress [1], [2]. Prolonged contamination disrupts the trophic structure of ecosystems and indirectly threatens predator populations as prey availability declines. Unregulated harvesting of prey species further amplifies this pressure and increases the risk of local extinction and ecosystem collapse [3], [4]. Mathematical modeling provides a framework for analyzing these coupled ecological stresses. The classical

Lotka–Volterra predator–prey equations describe interspecific interactions but do not capture pollutant effects or exploitation. Numerous extensions incorporate additional mechanisms, including toxicant-induced mortality, density-dependent harvesting, and nonlinear predator functional responses[5], [6], [7], [8].

Recent advances demonstrate the breadth of such approaches. For example, fractional-order formulations that include toxicity, prey refuge, and combined harvesting show that toxic stress and overharvesting can destabilize plankton–fish systems unless protective refuges are present [9]. Models that treat uncertain biological parameters with fuzzy optimization indicate that harvesting policies must be conservative under high toxicity to maintain both prey and predator populations [10]. Analyses of toxin-producing predator–prey systems with threshold-based harvesting reveal that spatial heterogeneity and inappropriate harvest thresholds can trigger pattern formation and local population collapse [11]. Similarly, incorporating Allee effects alongside harvesting effort highlights the risk of complex bifurcations and extinction when prey densities fall below critical levels [12].

Nevertheless, few models simultaneously integrate pollutant toxicity and prey harvesting in an aquatic context while explicitly identifying the parameter ranges that guarantee persistence of both species. This study addresses that gap by investigating the parameter ranges of toxicity and harvesting rates within which the interior equilibrium of a predator–prey system remains locally stable. A nonlinear system of differential equations is formulated, incorporating toxic stress on the prey population, the predator's functional response, and prey harvesting. The Routh–Hurwitz stability criterion offers a rigorous analytical approach to examine the local stability of equilibrium points in predator–prey systems. By converting nonlinear biological interactions into mathematical conditions, it uniquely facilitates the identification of critical thresholds that govern the persistence, extinction, or oscillation of species populations under varying ecological and environmental parameters. The analysis proceeds by determining all equilibrium points and applying the Routh–Hurwitz criterion [13], [14] to assess local stability. Numerical simulations are then used to demonstrate how changes in toxicity and harvesting parameters can shift the system from stable to unstable dynamics. The central hypothesis is that critical threshold values exist so that the equilibrium is maintained when the toxicity and harvesting rates stay below these limits and becomes unstable once they are exceeded. This quantitative framework is intended to inform sustainable water-resource management policies that balance ecological resilience with human exploitation.

2. RESEARCH METHOD

This research is a theoretical study designed to construct, modify, and analyze a mathematical model of predator–prey interactions in polluted aquatic ecosystems, incorporating the effects of harvesting and toxicity.

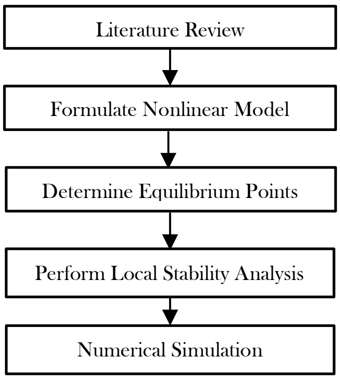

Figure 1. Research Process Stages

2.1 Literature Review

The procedure begins with an extensive literature review of classical predator–prey models, such as the Lotka–Volterra and Leslie–Gower frameworks, as well as subsequent studies that integrate harvesting and toxic influences in aquatic environments [15], [16]. The Lotka–Volterra model represents the most classical form of the predator–prey framework, developed independently by [17] and [18]. The model is expressed as follows:

$$\frac{dx}{dt} = \alpha x - \beta xy, \qquad \frac{dy}{dt} = \delta xy - \gamma y, \tag{1}$$

where $x(t)$ denotes the prey population, $y(t)$ the predator population, α the prey growth rate, β the predation rate, γ the predator mortality rate, and δ the efficiency with which prey biomass is converted into predator biomass. The model assumes that, in the absence of predators, the prey population grows exponentially, and that interactions between predators and prey drive changes in both populations. Although relatively simple, this framework successfully captures the periodic cycles between the two populations, even though it does not account for environmental carrying capacity or other ecological complexities.

Meanwhile, the Leslie-Gower model is a modification of the Lotka-Volterra framework introduced by [19], which incorporates the assumption that predator growth depends on the ratio of predators to prey. The model is expressed as follows:

$$\frac{dx}{dt} = x\left(1 - \frac{x}{K}\right) - \beta xy, \qquad \frac{dy}{dt} = y\left(-d + \frac{cx}{x + A}\right), \tag{2}$$

where K represents the carrying capacity of the prey population, and the ratio-dependent term in the predator equation moderates the effect of predation according to prey availability. This model is considered more realistic because it introduces an environmental (logistic) constraint on the prey and prevents the unbounded predator growth observed in the Lotka-Volterra model. Moreover, it allows for a broader range of positive equilibrium solutions that can remain stable.

In population modeling, harvesting refers to the removal of individuals from a population for economic purposes, such as fishing, hunting, or agriculture. Harvesting is typically incorporated into predator-prey models through control terms such as $-h_1x$ and/or $-h_2y$, which represent the harvesting rates of prey and predator populations, respectively. An example of a model with prey harvesting is given by:

$$\frac{dx}{dt} = x\left(1 - \frac{x}{K}\right) - \beta xy - h_1x. \tag{3}$$

Harvesting can help regulate populations and provide economic benefits, but it may also destabilize the ecosystem if not carefully managed. Therefore, evaluating both the biological and economic aspects of equilibrium, known as bionomic equilibrium, is essential when assessing such systems [15], [20].

In addition to harvesting, toxicity and environmental pollution represent critical factors that can disrupt population stability in aquatic ecosystems. Pollutants such as heavy metals, pesticides, and industrial waste may inhibit the growth of both prey and predator populations, reduce fertility, or even cause direct mortality. Within mathematical models, these toxic effects are often incorporated through additional terms that represent increased mortality rates or reduced reproductive efficiency. For example, the toxic impact on the prey population can be expressed by adding an extra mortality term to the prey equation:

$$\frac{dx}{dt} = x\left(1 - \frac{x}{K}\right) - \beta xy - \theta x^3, \tag{4}$$

where θx^3 represents a reduction in the prey population caused by pollution that exhibits saturating or bioaccumulative effects. Recent studies indicate that increasing toxicity can diminish ecosystem stability and may lead to the extinction of predator populations at a faster rate than that of their prey [21], [22].

The next stage involves formulating a mathematical model that accurately reflects the biological and ecological context, followed by a nondimensionalization process to simplify the system and facilitate stability analysis. Equilibrium points are determined by solving the stationary system ($dq/d\tau = 0, dw/d\tau = 0$), after which the Jacobian matrix is derived and the Routh-Hurwitz criterion is applied to assess local stability. All analytical steps are supported by MATLAB for both symbolic and numerical computations, and the stability results are further validated through numerical simulations of population dynamics under various parameter scenarios [23].

2.2 Mathematical Model Formulation

The development of this mathematical model is based on a number of ecological assumptions that represent the actual conditions of polluted aquatic ecosystems. First, the prey population (small fish or plankton) experiences logistic growth in the presence of environmental carrying capacity. Second, the predators (large fish) feed on the prey as their main food source. Third, there is harvesting activity of the prey with an exploitation rate considered proportional to the population size. Fourth, the aquatic environment is polluted by toxic substances (for example, heavy metals or industrial waste), which have a negative impact on the growth of both prey and predators. Fifth, the pollution is assumed to be constant and homogeneous within the environment [22], [16]. A study by [24] indicates that the system's stability is highly sensitive to the combined effects of toxicity and exploitation intensity, highlighting the need for resource management strategies that address these multidimensional interactions simultaneously.

To capture the combined effects of harvesting and toxic contamination on aquatic predator-prey interactions, the model begins with a system in which a prey population $Q(t)$ (e.g., plankton or small fish) coexists with a predator population $W(t)$ (e.g., larger fish) in a polluted aquatic environment. The prey grows logistically, is harvested, suffers toxic stress, and is consumed by predators. Predators depend on prey for food and are also affected by toxicity. Time t measures population changes in this system. Before scaling, the key parameters are

E : harvesting effort on prey (e.g., fishing intensity), $E \geq 0$
K_1 : environmental carrying capacity of prey (same units as Q)
r_1, r_2 : intrinsic growth rates of prey and predator, respectively (> 0).
a : predation rate coefficient (effectiveness of predators per unit prey)
b : catchability coefficient of prey by harvesting effort
c : toxicity coefficient on prey, modeled as proportional to Q^3
k : conversion efficiency of consumed prey into predator biomass
f : toxicity coefficient on predators, proportional to W^2
ϑ : efficiency with which prey biomass supports predator carrying capacity.

To simplify analysis and allow use of parameter estimates, scaling the variables as:

$$\dot{q} = \frac{Q}{K_1}, \qquad \dot{w} = \frac{W}{\vartheta Q}, \qquad \tau = r_1 t. \tag{5}$$

Substituting these into the prey equation of the original model and simplifying gives:

$$\frac{d\dot{q}}{d\tau} = \dot{q}(1-\dot{q}) - \left(\frac{aK_1}{r_1}\right)\vartheta\dot{q}^2 w - \left(\frac{bE}{r_2}\right)\dot{q} - \left(\frac{cK_1^2}{r_1}\right)\dot{q}^3. \tag{6}$$

For the predator equation, the same procedure yields:

$$\frac{d\dot{w}}{d\tau} = \left(\frac{r_2}{r_1}\right)\dot{w}(1-\dot{w}) + \left(\frac{kK_1}{r_1}\right)\dot{q}\dot{w} - \left(\frac{fK_1}{r_1}\right)\vartheta\dot{q}\dot{w}^2. \tag{7}$$

This model captures the complex interactions among biological processes (growth and predation), economic factors (harvesting), and environmental influences (toxicity). To investigate this sensitivity in a tractable way, a set of new dimensionless parameters is introduced, condensing the original biological and anthropogenic factors into a normalized form suitable for stability analysis and comparison with prior studies.

$$\theta = \frac{aK_1}{r_1}, \qquad \kappa = \frac{bE}{r_1}, \qquad \varphi = \frac{cK_1^2}{r_1}, \qquad \chi = \frac{r_2}{r_1}, \qquad \zeta = \frac{kK_1}{r_1}, \qquad \phi = \frac{fK_1}{r_1}, \tag{8}$$

the normalized predator–prey model becomes

$$\frac{d\dot{q}}{d\tau} = \dot{q}\,(1-\dot{q}\,) - \theta\vartheta\dot{q}^{\,2}\dot{w} - \kappa\dot{q} - \varphi\dot{q}^{\,3}, \qquad \frac{d\dot{w}}{d\tau} = \chi\dot{w}(1-\dot{w}\,) + \zeta\dot{q}\,\dot{w} - \phi\vartheta\dot{q}\dot{w} \tag{9}$$

Each new parameter combines biological and anthropogenic effects. There are the relative predation rate θ, the dimensionless harvesting pressure κ, the cubic toxicity impact on prey φ, the predator–prey growth-rate ratio χ, the predator growth contribution from prey consumption ζ, and the toxicity impact on predators ϕ.

All parameters are assumed to be positive and constant throughout the analysis. Numerical values for these parameters are obtained from literature studies and from conservative estimates based on fisheries biomathematics references [21], [24]. The first equation describes changes in the prey population influenced by logistic growth, predation, harvesting, and saturating toxicity. The second equation represents the dynamics of the predator population, which increases through predation on the prey but declines because of environmental toxicity during feeding. This system extends the classical predator–prey model by incorporating two key factors: prey harvesting and toxic effects, both of which affect the stability and persistence of populations within the ecosystem [25]. The model is analyzed through both analytical and numerical methods to identify conditions for local and global stability, and simulations are carried out to observe the system's long-term behavior under varying environmental parameters.

3. RESULT AND ANALYSIS

3.1 Equilibrium Points and Stability Analysis

An equilibrium point of a dynamical system occurs when $\frac{dx}{dt} = 0$, meaning the system experiences no change in state. In many cases, the system may have one or more equilibrium points, which can be stable (attractors), unstable (repellors), or semi-stable. Local stability is determined by examining the sign of the real parts of the eigenvalues of the Jacobian matrix evaluated at the equilibrium point. If all real parts are negative, the point is asymptotically stable [26]. To analyze the local stability of a nonlinear system, a commonly used technique is linearization, which approximates the nonlinear system by a linear one in the neighborhood of the equilibrium. This process involves computing the Jacobian matrix of the system, given by:

$$J_{ij} = \frac{\partial f_i}{\partial x_j}, \tag{10}$$

evaluated at the equilibrium point.

The linear system obtained through linearization exhibits local behavior that closely approximates the original nonlinear system near that point. Consequently, the stability properties can be determined from the eigenvalues of the Jacobian matrix. Although linearization does not guarantee global stability, this approach is highly effective and widely applied in population dynamics studies [25], [27].

Stability analysis is carried out by linearizing the nonlinear system around the equilibrium point and applying the Routh–Hurwitz criterion to determine whether the system will return to its original state after a small disturbance. This section discusses the derivation of the equilibrium points, the construction of the Jacobian matrix, and the application of the local stability criterion.

Determination of Equilibrium Points

The equilibrium points of the system are obtained by substituting the stationary conditions into the model, that is, by setting the time derivatives of each variable equal to zero. Based on the nondimensional model:

$$\begin{aligned} &\dot{q}\,(1-\dot{q}\;) - \theta\vartheta\dot{q}\,^2\dot{w} - \kappa\dot{q}\; - \varphi\dot{q}\,^3 = 0,\\ &\chi\dot{w}(1-\dot{w}\;) + \zeta\dot{q}\,\dot{w} - \phi\vartheta\dot{q}\dot{w}^2 = 0. \end{aligned} \tag{11}$$

Four candidate equilibrium points are obtained, namely both prey and predator populations are extinct $E_0 = (0,0)$, only the prey population persists without predators $E_1 = (\dot{q}_1, 0)$, only the predator population persists $E_2 = (0, \dot{w}_1)$, and the interior point where both prey and predator populations coexist positively $E^* = (\dot{q}^*, \dot{w}^*\,)$.

The points E_0, E_1 and E_2 can be obtained directly, whereas the interior point E^* is computed numerically because the resulting equations are nonlinear. As an illustration, the following parameter values are used:

Table 1. Parameter Values

$\boldsymbol{\theta}$	$\boldsymbol{\kappa}$	$\boldsymbol{\varphi}$	$\boldsymbol{\zeta}$	$\boldsymbol{\phi}$	$\boldsymbol{\vartheta}$	$\boldsymbol{\chi}$
$\mathbf{0.15}$	0.5	0.3	0.25	0.2	0.5	0.65

Linearization and Jacobian Matrix

To analyze local stability around an equilibrium point, the nonlinear system is linearized by means of the Jacobian matrix. For the two-variable system $(\dot{q}, \dot{w})$ the Jacobian is defined as:

$$J(\dot{q},\dot{w}) = \begin{pmatrix} \dfrac{\partial f_1}{\partial \dot{q}} & \dfrac{\partial f_1}{\partial \dot{w}} \\ \dfrac{\partial f_2}{\partial \dot{q}} & \dfrac{\partial f_2}{\partial \dot{w}} \end{pmatrix}. \tag{12}$$

where:

$$\begin{aligned} f_1(\dot{q},\dot{w}) &= \dot{q}(1-\dot{q}) - \theta\vartheta\dot{q}^2\dot{w} - \kappa\dot{q} - \varphi\dot{q}^3,\\ f_2(\dot{q},\dot{w}) &= \ \chi\dot{w}(1-\dot{w}) + \zeta\dot{q}\dot{w} - \phi\vartheta\dot{q}\dot{w}^2. \end{aligned} \tag{13}$$

From these expressions, the Jacobian matrix becomes

$$J(\dot{q},\dot{w}) = \begin{pmatrix} (1-2\dot{q}) - 2\theta\vartheta\dot{q}\dot{w} - \kappa \dot{-} 3\varphi\dot{q}^2 & -\theta\vartheta\dot{q}^2 \\ \zeta\dot{w} - \phi\vartheta\dot{w}^2 & \chi(1-2\dot{w}) + \zeta\dot{q} - 2\phi\vartheta\dot{q}\dot{w} \end{pmatrix}. \tag{14}$$

This matrix is then evaluated at each equilibrium point and used with the Routh–Hurwitz criterion to determine local stability.

Routh–Hurwitz Stability Analysis

For a two-dimensional system, local stability can be determined from the characteristic equation of the Jacobian matrix using the Routh–Hurwitz criterion. This criterion is an algebraic method employed to assess the local stability of a linear dynamical system without explicitly calculating the roots of the characteristic polynomial. It is particularly useful for two- or three-dimensional systems, as it provides stability conditions based solely on the signs of the coefficients of the characteristic polynomial. Suppose a linear system has the following characteristic equation:

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0. \tag{15}$$

A system is considered stable if and only if all roots of the characteristic polynomial have negative real parts, or equivalently, if all the coefficients satisfy the Routh–Hurwitz conditions (being positive and not producing any sign changes). For a second-order system:

$$\lambda^2 + a_1\lambda + a_0 = 0, \tag{16}$$

the Routh–Hurwitz criterion states that the system is stable if and only if $a_1 > 0$ and $a_0 > 0$. This criterion can be extended to higher-order systems by constructing the Routh–Hurwitz table, which is organized using combinations of the polynomial coefficients. The advantage of this method lies in its efficiency, as it provides information on system stability without the need to explicitly solve the characteristic equation [23], [28].

Stability analysis is carried out by examining the trace and determinant of the Jacobian matrix evaluated at the equilibrium points obtained numerically. According to the Routh–Hurwitz criterion for a two-dimensional system, the interior equilibrium point E^* is locally stable for the characteristic equation of the Jacobian matrix:

$$\lambda^2 - Tr(J)\lambda + det\,(J) = 0, \tag{17}$$

if the following two conditions are satisfied: (i) the trace of the Jacobian matrix is negative ($Tr(J) < 0$), and (ii) the determinant is positive ($det(J) > 0$). A negative trace indicates that the average rate of change in the system tends to decrease, representing damping in the dynamics toward the equilibrium point. A positive determinant ensures that no divergent direction exists, so the equilibrium point is not a saddle point. Conversely, if $det(J) < 0$, the equilibrium becomes a saddle point and is unstable, even if the trace is negative [29]. Satisfying these conditions ensures that all eigenvalues of the system have negative real parts, meaning the system tends to return to equilibrium after small perturbations.

Before substituting numerical values, the explicit forms of the Jacobian's trace and determinant are derived to facilitate the analytical stability assessment. For the two-dimensional system, the trace is given by

$$Tr(J) = \left((1 - 2\dot{q}) - 2\theta\vartheta\dot{q}\dot{w} - \kappa \dot{-} 3\varphi\dot{q}^2\right) + (\,\chi(1 - 2\dot{w}) + \zeta\dot{q} - 2\phi\vartheta\dot{q}\dot{w}),$$

and the determinant is expressed as

$$\det(J) = \left((1 - 2\dot{q}) - 2\theta\vartheta\dot{q}\dot{w} - \kappa \dot{-} 3\varphi\dot{q}^2\right)(\chi(1 - 2\dot{w}) + \zeta\dot{q} - 2\phi\vartheta\dot{q}\dot{w}) + \theta\vartheta\dot{q}^2(\zeta\dot{w} - \phi\vartheta\dot{w}^2).$$

These symbolic expressions are then evaluated at each equilibrium point $(\dot{q}^*, \dot{w}^*)$ to obtain $Tr(J)$ and $\det(J)$, which are subsequently used in the Routh–Hurwitz conditions ($Tr(J) < 0, \det(J) > 0$) to determine the local stability of the system.

The Routh–Hurwitz criterion is widely employed in biomathematics and engineering to evaluate the stability of mathematical models, including predator–prey systems, population control, and epidemiological models [22]. Its systematic and algebraic nature provides a robust framework for understanding how parameter variations influence the stability and behaviour of nonlinear dynamical systems.

3.2 Numerical Simulations

Numerical simulations are conducted to visualize the dynamics of the predator–prey system in a polluted aquatic ecosystem, as well as to confirm the analytical results obtained through the Routh–Hurwitz approach. Specifically, these simulations aim to: 1) show the evolution of prey and predator populations over time, 2) analyse the local stability of the interior equilibrium point, 3) examine the effects of variations in the toxicity index (φ) and prey harvesting rate (κ) on population dynamics, 4) assess the sustainability of prey and predator populations under conditions of environmental pollution and exploitation.

In this study, the parameters κ (harvesting intensity), φ (toxicity level), and ϑ (predator conversion efficiency) are each restricted to the interval $[0, 1]$ to represent moderate ecological pressures. This range prevents either harvesting or pollution from dominating the system dynamics in an unrealistic way. Within these bounds, κ is fixed at 0.5, reflecting a harvesting rate well below the upper limit and consistent with sustainable exploitation levels recommended by [31]. Likewise, φ is set at 0.3 to capture measurable but non-catastrophic toxic effects, avoiding the extreme assumptions (such as $\kappa \geq 1$) found in some earlier studies. These choices help ensure that population decline in the model is not driven solely by excessive anthropogenic stress.

The parameter ϑ, representing the efficiency with which prey biomass contributes to predator carrying capacity, is likewise taken as 0.5, indicating that predators can convert roughly half of the available prey biomass into their own population support. This value balances biological realism, where energy transfer between trophic levels is rarely 100%, with the need to model a functioning predator–prey interaction. Overall, the chosen values $\varphi = 0.3$, $\kappa = 0.5$, and $\vartheta = 0.5$ lie comfortably within the designated moderate range, aligning with the conservative management principles of [31] and supported by findings such as [32], which show that excessive toxicity disrupts ecosystem equilibrium.

Table 2. Simulation Parameters

Parameters	Values	Descriptions	Sources
θ	0.15	Rate of predator–prey interaction	[30]
κ	0.5	Exploitation rate of prey due to harvesting	Assumption
φ	0.3	Toxicity coefficient for the prey	Assumption
χ	0.65	Ratio of the natural growth rate of predators to the natural growth rate of prey	[30]
ζ	0.25	Predator growth efficiency	[30]
ϕ	0.2	Predator sensitivity to toxicity	[30]
ϑ	0.5	Conversion efficiency coefficient of prey into predator carrying capacity	Assumption

After defining the main parameters that constitute the model system, numerical simulations are carried out using the parameter values presented in Table 2. These parameters reflect the influence of biological interactions between prey and predator, as well as external pressures in the form of environmental toxicity and prey harvesting.

Simulation of Equilibrium Point Stability

Numerical simulations are performed by solving the system of nonlinear differential equations using the fourth-order Runge–Kutta method implemented in MATLAB R2023b's ode45 function. The solver employed a relative tolerance of 1×10^{-6} and an absolute tolerance of 1×10^{-8}, and runs were performed for various initial population conditions. The initial conditions tested include $n_1 = (1.5,1), n_2 = (1.5,0.6)$, and $n_3 = (1.5,2)$ in order to observe population dynamics under different initialization scenarios.

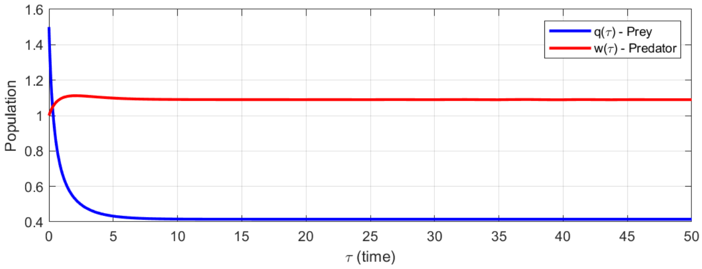

Figure 2. Population Dynamics for $n_2 = (1.5,1)$

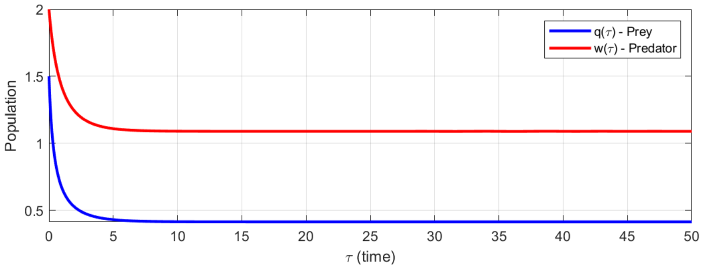

Figure 3. Population Dynamics for $n_3 = (1.5,2)$

Figure 2 illustrates the temporal dynamics of the prey population $\dot{q}(\tau)$ and the predator population $\dot{w}(\tau)$ over the nondimensional time interval $\tau \in [0,50]$. It is observed that the prey population (blue line) experiences a sharp decline at the beginning and then stabilizes at a low value. In contrast, the predator population (red line) shows a slight initial increase before reaching a steady state close to its initial value. The drastic decline in prey is likely due to the combined pressures of pollution and harvesting, which inhibit the prey's reproduction rate. Nevertheless, the predator population remains stable at a relatively high level, possibly due to efficient prey-to-predator conversion or adaptive capacity under toxic conditions. This pattern is consistent with the local stability analysis, which indicates that the system approaches a stable equilibrium point following the initial fluctuations.

The blue curve ($\dot{q}(\tau)$) in Figure 3 shows that the prey population rapidly declines from a low initial value and then stabilizes around 0.5. Meanwhile, the red curve ($\dot{w}(\tau)$) indicates that the predator population starts at a high value (around 2) and gradually decreases until it reaches a steady-state near 1. This behaviour reflects a predator–prey system in which an initially abundant predator population exerts strong pressure on the prey, causing an immediate sharp decline in prey numbers. As the prey population decreases, the predator population also diminishes due to limited food resources. After some time, both populations reach a steady state, with the prey persisting at a low stable level and the predator maintaining a moderate population. Compared to the previous simulation, the different initial conditions produce distinct dynamics. While in the first simulation the predator population increased due to abundant prey, in this case the predator population decreases initially because it is excessively large relative to the available prey. Nevertheless, both systems demonstrate convergence toward the equilibrium point.

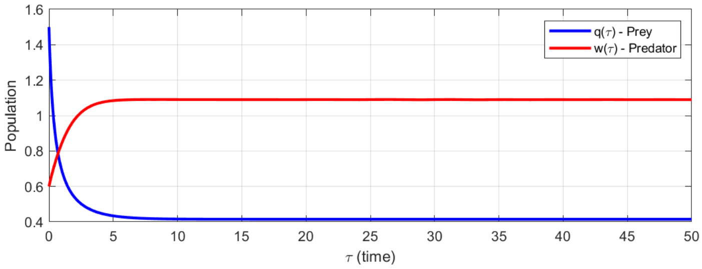

Figure 4. Population Dynamics for $n_2 = (1.5, 0.6)$

Figure 4 illustrates the classical predator–prey interaction mechanism. In the initial phase, the abundant prey population causes a significant increase in the predator population. However, the rising number of predators exerts high pressure on the prey population, leading to a drastic decline in prey abundance. Following this decline, the growth rate of the predator population slows until a steady-state condition is reached. This phenomenon indicates the existence of an ecological equilibrium point, where both populations can persist at stable numbers without causing the extinction of either species. The model aligns with theoretical predictions, which suggest that predator–prey systems tend to achieve dynamic equilibrium after initial fluctuations.

To obtain the interior equilibrium point of the modelled predator–prey system, a numerical method for solving nonlinear systems was employed using the Newton–Raphson approach, implemented via the solve function in MATLAB. The search for the equilibrium point was conducted using various initial guesses: $(0.2, 0.2)$, $(0.5, 0.5)$, $(0.2, 0.8)$ and $(0.3, 0.8)$. These pairs were selected to represent different possible initial positions of the prey and predator populations in the phase space, encompassing both low and high population conditions. The numerical procedure was executed with a convergence tolerance of 1×10^{-8} and a maximum iteration limit of 100. The tolerance value was chosen to ensure precision up to eight decimal places, while the iteration limit was set to prevent unbounded computations in the event of solution divergence.

In initial guess $(0.2, 0.2)$, the simulation produced an equilibrium point at $(\dot{q}^*, \dot{w}^*) = (0, 0)$, indicating total extinction of both prey and predator populations. The trace and determinant of the Jacobian matrix at this point are: $Tr(J) = 1.1500$, and $det\ (J) = 0.3250$. Since the trace is positive, this point does not satisfy the local stability conditions and is therefore unstable. This implies that the system will not remain at total extinction and will tend to move away from this point if perturbed. In initial guess $(0.5, 0.5)$, the simulation yielded an equilibrium point at $(\dot{q}^*, \dot{w}^*) = (0.4415, 0)$, indicating that only the prey population persists while the predator goes extinct. The Jacobian analysis gives: $Tr(J) = 0.2019$, and $det(J) = -0.4247$. Although the trace is positive, the negative determinant indicates conflicting growth and decay directions, making this point a saddle point and therefore unstable. In initial guess (0.2,0.8), the simulation produced an equilibrium point at $(\dot{q}^*, \dot{w}^*) = (0, 1)$. The Jacobian evaluated at this point yields: $Tr(J) = -0.1500$, and $det(J) = -0.3250$. The negative determinant indicates that this equilibrium is a saddle point. Despite the negative trace, the negative determinant ensures the presence of divergent directions, so the system cannot persist at this point. This point represents a situation where the prey is extinct but the predator survives, which is ecologically unrealistic because predators cannot sustain themselves without prey as a food source. The stability analysis supports this interpretation, as the saddle-point nature indicates that the system will move away from this point toward a more balanced condition, such as the interior equilibrium where both populations can coexist. Finally, in initial guess (0.3,0.8), the simulation produced an interior equilibrium point at $(\dot{q}^*, \dot{w}^*) = (0.4146, 1.0899)$, where both prey and predator populations coexist. The Jacobian analysis yields: $Tr(J) = -1.3052$, and $det(J) = 0.4177$. These values satisfy the Routh–Hurwitz stability criteria ($Tr(J) < 0$ and $det(J) > 0$), indicating that this equilibrium point is locally stable. This suggests that the system will return to this point after small perturbations, allowing both prey and predator populations to persist together over the long term. The above explanation is presented in the following table.

Table 3. Stability Analysis of Equilibrium Points.

No.	Equilibrium Points	Coordinates	$Tr(J)$	$det(J)$	Stability Verdict
1.	E_0 (total extinction)	(0,0)	1.1500	0.3250	Unstable ($Tr > 0$)
2.	E_1 (prey only)	(0.4415,0)	0.2019	-0.4247	Unstable-saddle ($det < 0$)
3.	E_2 (predator only)	(0,1)	-0.1500	-0.3250	Unstable-saddle ($det < 0$)
4.	E^* (interior coexistence)	(0.4146,1.0899)	-1.3052	0.4177	Locally asymptotically stable ($Tr < 0, det > 0$)

The boundary equilibria E_0, E_1, and E_2 are all unstable and carry distinct ecological interpretations. The total extinction state E_0, while mathematically feasible, behaves as a source such that small perturbations drive the system away from extinction; ecologically, this suggests that complete collapse is not a stable long-term outcome under the modeled conditions. The prey-only equilibrium E_1 manifests as a saddle point, representing a fragile state in which prey can persist without predators, but this balance is easily disrupted. Any reintroduction of predators or environmental variation can destabilize the system, pushing it toward coexistence or collapse. Finally, the predator-only equilibrium E_2 is both unstable and biologically unrealistic, since predators cannot persist indefinitely in the absence of prey. The instability of this saddle point highlights that predator-only survival is transient and dependent on external forcing or alternative food sources.

Only solutions that meet the specified tolerance criteria and yield positive population values $(\dot{q}^* > 0, \dot{w}^* > 0)$ are accepted as valid equilibrium points. The results indicate that only the interior equilibrium, where both populations coexist, is stable, whereas equilibrium points corresponding to total extinction, predator extinction, or prey extinction are unstable. These findings emphasize the importance of maintaining environmental conditions and system parameters within ranges that support the simultaneous existence of both populations, ensuring the sustainability of polluted aquatic ecosystems.

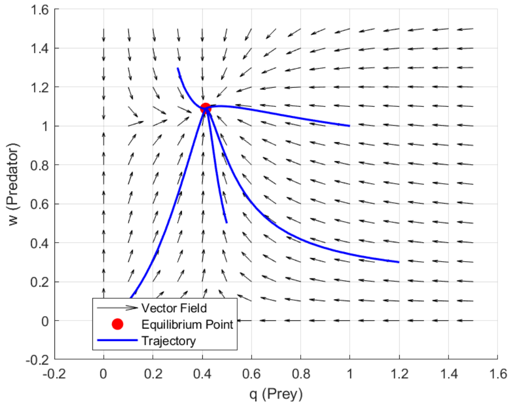

Figure 5. Phase portrait with vector field and trajectories

For the phase portrait below, five representative initial conditions were selected to visualize the system's trajectories: $(\dot{q}(0), \dot{w}(0)) = (0.1, 0.1), (0.5, 0.5), (1, 1), (1.2, 0.3)$, and $(0.3, 1.3)$. These initial points demonstrate how different starting populations of prey and predator evolve over time toward the interior equilibrium.

Figure 5 presents the phase map of the predator–prey system in a polluted aquatic ecosystem, visualized in the phase space $(\dot{q}, \dot{w})$, where $\dot{q}$ represents the prey population and $\dot{w}$ represents the predator population. The vector field (shown as black arrows) illustrates the rate of change of the populations at each point in the state space, while the blue lines depict the system trajectories from various initial conditions toward the equilibrium point. The red point on the graph represents the interior equilibrium obtained at $(\dot{q}, \dot{w}) = (0.4146, 1.0899)$. Based on the linear analysis using the trace and determinant of the Jacobian matrix ($Tr(J) = -1.3052$, and $det(J) = 0.4177$), this point is locally asymptotically stable. This is evident from the trajectories, all of which consistently move toward and converge at this equilibrium, regardless of their initial positions.

Ecologically, the stability of this equilibrium point indicates that, under moderate levels of pollution and harvesting, the ecosystem can continue to sustain both populations. Although prey and predator numbers may fluctuate initially, they eventually converge to a balanced state. Using representative parameter values similar to those reported in [24], a Routh–Hurwitz analysis shows that the interior equilibrium remains locally stable as long as the combined toxicity and harvesting parameter ϑ stays below approximately $4.1\text{–}4.2\ day^{-1}$. Once this threshold is exceeded, the determinant of the Jacobian becomes negative, signalling a loss of local stability and the onset of large oscillations or possible extinction of one or both species. These results reinforce the analytical predictions of the Routh–Hurwitz criteria and align with previous studies indicating that predator–prey systems exposed to external pressures such as pollution and exploitation activities can remain stable provided key parameters remain within defined limits [24].

3.3 Discussion

Analytical evaluation using the Routh–Hurwitz criteria shows that the stability of the predator–prey system is highly sensitive to the toxicity parameter (ϑ) and the prey harvesting rate (κ). These parameters define clear thresholds separating stable coexistence from ecological collapse. Numerical simulations support the analytical predictions, revealing four equilibrium points: $(0,0)$, $(0.4415,0)$, $(0,1)$, and $(0.4146,1.0899)$. Among these, only the interior point $(0.4146,1.0899)$ satisfies $Tr(J) < 0$ and $det(J) > 0$, confirming local stability. The other equilibria display either a positive trace or a negative determinant, indicating instability and representing ecological scenarios in which prey depletion prevents predator persistence.

The close agreement between analytical and numerical results strengthens confidence in the model's predictive capability. Ecologically, the findings demonstrate that balanced coexistence of predator and prey is possible when pollution and harvesting pressures remain within moderate limits. Exceeding these limits drives the system toward oscillations or extinction, underscoring the prey population's critical role in maintaining ecosystem resilience.

From a management perspective, the model provides quantitative guidance for sustainable policy. It identifies approximate critical values of combined toxicity and harvesting beyond which stability is lost. These thresholds can be translated into practical actions such as setting maximum pollutant loads, defining industrial discharge limits, and establishing harvest quotas to help maintain water quality and prevent overexploitation of prey species. Incorporating these scientific limits into fisheries regulations and pollution-control programs ensures that human activities remain within the ecosystem's capacity for self-renewal.

Overall, the consistency between analytical and numerical approaches, together with the clear ecological interpretation of each equilibrium state, highlights the model's value as a decision-support tool. By linking mathematical stability analysis with practical management targets, the study offers a robust framework for protecting aquatic ecosystems from the combined pressures of toxicity and exploitation.

4. CONCLUSION

This study examined the population dynamics of predator–prey interactions in a polluted aquatic ecosystem, incorporating the combined effects of toxicity and prey harvesting. The model was formulated as a two-dimensional nonlinear differential equation system and analysed using local stability methods based on the Routh–Hurwitz criteria, supported by numerical simulations. Both analytical and numerical results indicate that the interior equilibrium becomes unstable when toxicity and harvesting pressures exceed moderate levels. This condition occurs when the Jacobian matrix shows a negative determinant, and simulations reveal a sharp decline in the prey population followed by a predator collapse. No limit cycles or closed orbits were observed in either the time-series plots or the phase plane, implying that long-term coexistence cannot be maintained under uncontrolled pollution and harvesting. From a computational modelling perspective, the clear threshold behaviours identified highlights the value of such models as early-warning tools. Specifically, this framework can be integrated into decision-support systems (DSS) for fisheries or environmental management, where real-time or scenario-based simulations could alert policymakers when intervention thresholds are approached.

Despite these insights, several limitations should be acknowledged. The model uses fixed parameter values and assumes spatial homogeneity, constant environmental conditions, and no seasonal variability, which may oversimplify real aquatic ecosystems. It also omits factors such as prey refuges, adaptive predator behaviours, and biological resistance to toxins. Furthermore, the absence of empirical calibration means that the parameter ranges are illustrative rather than site-specific.

Future research can address these limitations in several concrete ways. First, incorporating spatial heterogeneity and diffusion terms would allow exploration of pollutant transport and habitat fragmentation. Second, adding time-dependent control parameters or optimal harvesting strategies could capture seasonal management interventions. Third, integrating prey refuge mechanisms, toxin degradation processes, or predator adaptation would provide a more realistic ecological response to pollution. Finally, calibrating the model with field data from specific aquatic environments would enable quantitative predictions and strengthen its utility for management and policy applications.

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